Chapter Seven

Rotation of a rigid object about a fixed axis

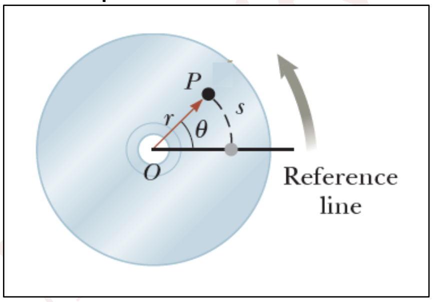
A **rigid object** is one that is non-deformable; that is, the relative locations of all particles of which the object is composed remain constant.

A **fixed axis** means that the rotation occurs about an axis that does not move, called the axis of rotation.

7-1- Angular Position, Velocity, and Acceleration

The **angular position** of a rigid object is defined as the angle θ between a reference line (OP) attached to the object and a reference line fixed in space.

$$\theta = \frac{s}{r} \qquad \text{.....(1)}$$



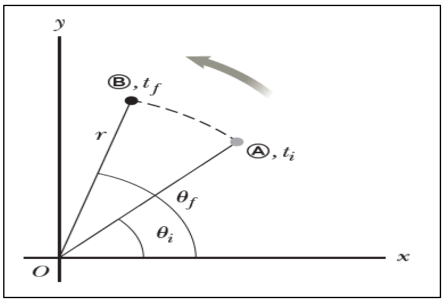
Where, s is the arc length.

The unit of θ is radian (rad).

The radian, being the ratio of two lengths, is a pure number and thus has no dimension.

As the particle in the rigid object travels from position Ⓐ to position Ⓑ in a time interval Δt, the reference line fixed to the object sweeps out an angle $\Delta\theta = \theta_f - \theta_i$. This quantity $\Delta\theta$ is defined as the **angular displacement** of the rigid object:

$$\Delta\theta = \theta_f - \theta_i \qquad \text{.....(2)}$$



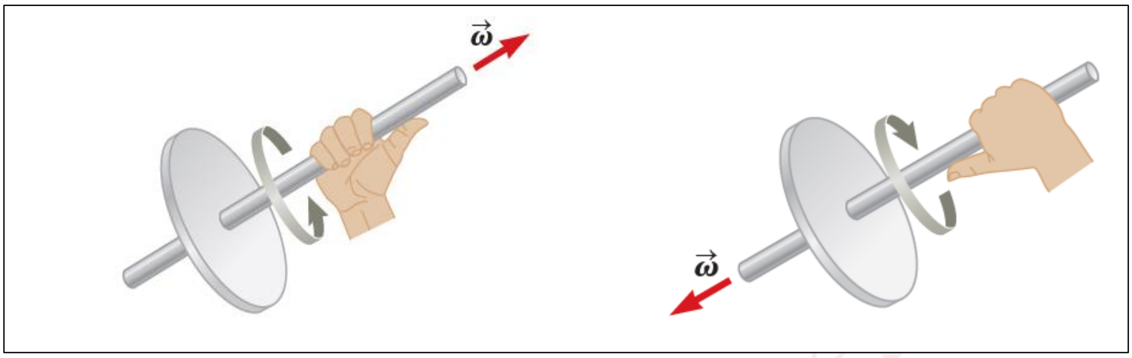
The **average angular speed** ω_{ave} is defined as the ratio of the angular displacement of a rigid object to the time interval Δt during which the displacement occurs:

$$\omega_{ave} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad \text{.....(3)}$$

The **instantaneous angular speed** ω is defined as the limit of the average angular speed as Δt approaches zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{......(4)}$$

We take ω to be positive when θ is increasing (counterclockwise motion) and negative when θ is decreasing (clockwise motion).
Angular speed has units of radians per second (rad/s), which can be written as s^{-1} because radians are not dimensional.



The **average angular acceleration** α_{ave} of a rotating rigid object is defined as the ratio of the change in the angular speed to the time interval Δt during which the change in the angular speed occurs:

$$\alpha_{ave} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad \text{.....(5)}$$

The **instantaneous angular acceleration** is defined as the limit of the average angular acceleration as Δt approaches zero:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} \qquad \text{......(6)}$$

When a rigid object is rotating about a fixed axis, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration. Therefore, the quantities θ, ω and α characterize the rotational motion of the entire rigid object as well as individual particles in the object.
In rotational motion, if the angular acceleration α is positive, then the angular velocity ω is increasing; if is negative, then is decreasing.

Ex: The angular position of a 0.36 m diameter flywheel is given by $\theta(t) = (2t^3)\,\text{rad}$, where t in seconds. (a) Find θ in radians and in degrees, at t_1= 2.0 s and t_2= 5.0 s (b) Find the distance that a particle on the flywheel rim moves over the time interval from t_1= 2.0 s to t_2= 5.0 s (c) Find the average angular velocity in rad/s and in rev/min, over that interval. (d) Find the instantaneous angular velocities at t_1= 2.0 s and t_2= 5.0 s.

Soln:

a)

$$\theta_1 = (2.0\ \text{rad/s}^3)(2.0\ \text{s})^3 = 16\ \text{rad}$$

$$= (16\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 920^\circ$$

$$\theta_2 = (2.0\ \text{rad/s}^3)(5.0\ \text{s})^3 = 250\ \text{rad}$$

$$= (250\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 14{,}000^\circ$$

b)

$$s = r\theta_2 - r\theta_1 = r\Delta\theta = (0.18\ \text{m})(234\ \text{rad}) = 42\ \text{m}$$

c)

$$\omega_{ave} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\ \text{rad} - 16\ \text{rad}}{5.0\ \text{s} - 2.0\ \text{s}} = 78\ \text{rad/s}$$

$$= \left(78\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 740\ \text{rev/min}$$

D)

$$\omega_1 = (6.0\ \text{rad/s}^3)(2.0\ \text{s})^2 = 24\ \text{rad/s}$$

$$\omega_2 = (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}$$

Ex: A child's top is spun with angular acceleration $\alpha(t) = (5t^3 - 4t)\text{rad/s}^2$. At t=0, the top has angular velocity 5 rad/s, and a reference line on it is at angular position θ=2 rad. A) Obtain an expression for the angular velocity ω(t) of the top. B) Obtain an expression for the angular position θ(t) of the top.

Soln:

A)

$$\alpha = \frac{d\omega}{dt} \rightarrow d\omega = \alpha dt \rightarrow \omega = \int \alpha dt = \int (5t^3 - 4t)dt = 5\frac{t^4}{4} - 4\frac{t^2}{2} + c \rightarrow \omega(t) = \frac{5}{4}t^4 - 2t^2 + c$$

At t=0 → ω=5 rad/s

$$\omega(t=0) = 5 = \frac{5}{4}(0)^4 - 2(0)^2 + c \rightarrow c = 5\ \text{rad/s}$$

$$\omega(t) = \left(\frac{5}{4}t^4 - 2t^2 + 5\right)\text{rad/s}$$

B)

$$\omega = \frac{d\theta}{dt} \rightarrow d\theta = \omega dt \rightarrow \theta = \int \omega dt = \int\left(\frac{5}{4}t^4 - 2t^2 + 5\right)dt = \left(\frac{5}{4}\right)\frac{t^5}{5} - 2\frac{t^3}{3} + 5t + c'$$

$$\theta(t) = \frac{t^5}{4} - \frac{2}{3}t^3 + 5t + c'$$

At t=0 → θ=2 rad

$$\theta(t=0) = 2 = \frac{(0)^5}{4} - \frac{2}{3}(0)^3 + 5(0) + c' \quad \rightarrow \quad c' = 2\,\text{rad}$$

$$\theta(t) = \left(\frac{t^5}{4} - \frac{2}{3}t^3 + 5t + 2\right)\text{rad}$$

7-2- Rigid object under constant angular acceleration:

If the angular acceleration of a rigid object is constant, then the average angular acceleration is equal to instantaneous angular acceleration, i.e

$$\alpha_{ave} = \alpha$$

If we take t_i=0 and t_f to be any later time t, we find that

$$\alpha_{ave} = \frac{\omega_f - \omega_i}{t - 0}$$

$$\omega_f = \omega_i + \alpha t \qquad \text{(for constant } \alpha\text{)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

Because angular speed at constant angular acceleration varies linearly in time, we can express the average angular speed in any time interval as

$$\omega_{ave} = \frac{\omega_f + \omega_i}{2} \qquad \text{(for constant } \alpha\text{)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}$$

Recalling that $\omega_{ave} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\theta_f - \theta_i}{t}$

Equating the above equation with equation (8), we get:

$$\theta_f - \theta_i = \frac{1}{2}(\omega_f + \omega_i)t \qquad \text{(for constant } \alpha\text{)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(9)}$$

We can obtain another useful expression for angular displacement at constant acceleration by substituting equation (7) into equation (9): $\theta_f - \theta_i = \frac{1}{2}(2\omega_i + \alpha t)t$

$$\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2 \qquad \text{(for constant } \alpha\text{)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}$$

Finally, we can obtain an expression for the final angular speed that does not contain a time interval by substituting the value of "t" from equation (7) into equation (9):

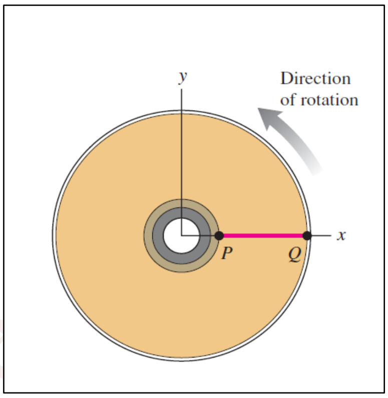
$$\theta_f - \theta_i = \frac{1}{2}(\omega_f + \omega_i)\left(\frac{\omega_f - \omega_i}{\alpha}\right)$$

$$\theta_f - \theta_i = \left(\frac{\omega_f^2 - \omega_i^2}{2\alpha}\right)$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \qquad \text{(for constant } \alpha\text{)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(11)}$$

Ex: A disc is slowing to a stop. The disc's angular velocity at t=0 is 27.5 rad/sec and its angular acceleration is a constant -10 rad/s^2. A line *PQ* on the disc's surface lies along the +x axis at t=0. (a) What is the disc's angular velocity at t=0.30 s (b) What angle does the line *PQ* make with the +x axis at this time?

Soln



a) $\omega_f = \omega_i + \alpha t \rightarrow \omega_f = 27.5 \text{ rad/s} + (-10 \text{ rad/s}^2)(0.3 \text{ s}) = 24.5 \text{ rad/s}$

b) $\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$

$$\theta_f = 0 + (27.5 \text{ rad/s})(0.3 \text{ s}) + \frac{1}{2}(-10 \text{ rad/s}^2)(0.3 \text{ s})^2 = 7.8 \text{ rad}$$

$$7.8 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 1.24 \text{ rev}$$

1.24 rev=(1+0.24)rev=(360^o+87^o)

Hence the line *PQ* makes an angle of 87^o with +x axis

Ex: A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. (A) If the angular speed of the wheel is 2.0 rad/s at t_i=0, through what angular displacement does the wheel rotate in 2.0 s? (B) Through how many revolutions has the wheel turned during this time interval? (C) What is the angular speed of the wheel at t= 2.0 s?
(D) Suppose a particle moves along a straight line with a constant acceleration of 3.50 m/s^2. If the velocity of the particle is 2.0 m/s at t_i= 0, through what displacement does the particle move in 2.0 s? What is the velocity of the particle at t= 2.0 s?

Soln.

A) $\Delta\theta = \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$

$$\Delta\theta = (2 \text{ rad/s})(2 \text{ s}) + \frac{1}{2}(3.50 \text{ rad/s}^2)(2 \text{ s})^2 = 11 \text{ rad} = 11 \text{ rad}\left(\frac{180^o}{\pi \text{ rad}}\right) = 630^o$$

B) $\Delta\theta = 630^o\left(\frac{1 \text{ rev}}{360^o}\right) = 1.75 \text{ rev}$

C) $\omega_f = \omega_i + \alpha t \rightarrow \omega_f = 2 \text{ rad/s} + (3.5 \text{ rad/s}^2)(2 \text{ s}) = 9 \text{ rad/s}$

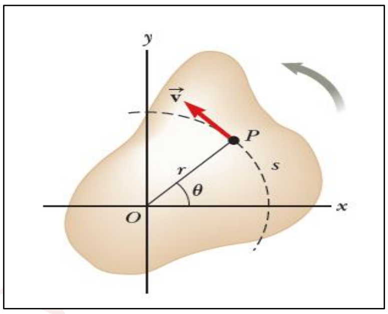
D) $\Delta x = x_f - x_i = v_i t + \frac{1}{2}at^2 \rightarrow \Delta x = (2 \text{ m/s})(2 \text{ s}) + \frac{1}{2}(3.50 \text{ m/s}^2)(2 \text{ s})^2 = 11 \text{ m}$

$$v_f = v_i + at \rightarrow v_f = 2 \text{ m/s} + (3.5 \text{ m/s}^2)(2 \text{ s}) = 9 \text{ m/s}$$

7-3- Angular and translational quantities:

In this section, we derive some useful relationships between the angular speed and acceleration of a rotating rigid object and the translational speed and acceleration of a point in the object.

When a rigid object rotates about a fixed axis as in figure below, every particle of the object moves in a circle whose center is on the axis of rotation. Because point *P* moves in a circle, the translational velocity vector $\vec{v}$ is always tangent to the circular path and hence is called *tangential velocity.* The magnitude of the tangential velocity of the point *P* is by definition the tangential speed.



$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$v = r\omega \qquad \text{……(12)}$$

The tangential speed of a point on a rotating rigid object equals the perpendicular distance of that point from the axis of rotation multiplied by the angular speed.

Therefore, in rotational motion of rigid object about a fixed axis, although every point on the rigid object has the same angular speed, but not every point has the same tangential speed because *r* is not the same for all points on the object.

The angular acceleration of the rotating rigid object is related to the tangential acceleration of the point *P* :

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$a_t = r\alpha \qquad \text{……(13)}$$

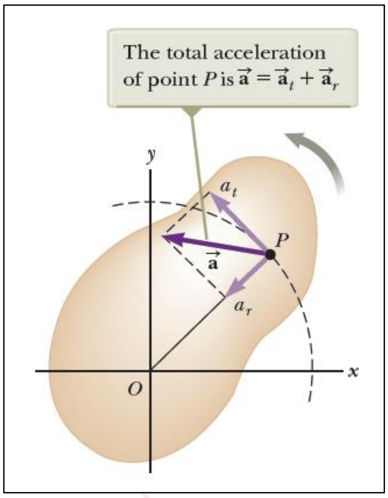
The tangential component of the translational acceleration of a point on a rotating rigid object equals the point's perpendicular distance from the axis of rotation multiplied by the angular acceleration.

We have seen, that a point moving in a circular path undergoes a radial acceleration a_r directed toward the center of rotation and whose magnitude is that of the centripetal acceleration

$$a_r = \frac{v^2}{r}$$

$$v = r\omega$$

$$a_r = r\omega^2 \qquad \text{……(14)}$$



The total acceleration vector at the point is:

$$\vec{a} = \vec{a}_t + \vec{a}_r \qquad \text{......(15)}$$

The magnitude of $\vec{a}$ at the point *P* on the rotating rigid object is:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^4 + \omega^4} \qquad \text{......(16)}$$

Ex: A disk 8 cm in radius rotates at a constant rate of 1200 rev/min about its central axis. Determine (a) its angular speed, (b) the tangential speed at a point 3 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2 s.

Soln:

(a) $$\omega = \frac{2\pi\,\text{rad}}{1\,\text{rev}}\left(\frac{1\,200\ \text{rev}}{60.0\ \text{s}}\right) = \boxed{126\ \text{rad/s}}$$

(b) $$v = \omega r = (126\ \text{rad/s})(3.00\times10^{-2}\ \text{m}) = \boxed{3.77\ \text{m/s}}$$

(c) $$a_c = \omega^2 r = (126)^2(8.00\times10^{-2}) = \boxed{1\,260\ \text{m/s}^2}$$

(d) $$s = r\theta = \omega r t = (126\ \text{rad/s})(8.00\times10^{-2}\ \text{m})(2.00\ \text{s}) = \boxed{20.1\ \text{m}}$$

Ex: A wheel 2 m in diameter lies in a vertical plane and rotates with a constant angular acceleration of 4 rad/s^2. The wheel starts at rest at t=0, and the radius vector of a certain point *P* on the rim makes an angle of 57.3° with the horizontal at this time. At *t* =2 s, find (a) the angular speed of the wheel, (b) the tangential speed and the total acceleration of the point *P*, and (c) the angular position of the point *P.*

Soln:

a) $\omega_f = \omega_i + \alpha t \rightarrow \omega_f = 0 + (4)(2) = 8\ \text{rad/s} \rightarrow \omega_f = 0 + (4)(2) = 8\ \text{rad/s}$

b)

$$v = r\omega = 1.00\ \text{m}\left(8.00\ \text{rad/s}\right) = 8.00\ \text{m/s}$$

$$a_c = r\omega^2 = 1.00\ \text{m}\left(8.00\ \text{rad/s}\right)^2 = 64.0\ \text{m/s}^2$$

$$a_t = r\alpha = 1.00\ \text{m}\left(4.00\ \text{rad/s}^2\right) = 4.00\ \text{m/s}^2$$

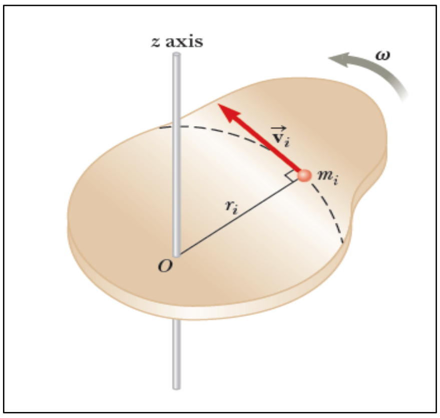
$$a = \sqrt{a_c^2 + a_t^2} = \sqrt{\left(64.0\ \text{m/s}^2\right) + \left(4.00\ \text{m/s}^2\right)} = 64.1\ \text{m/s}^2$$

$$\phi = \tan^{-1}\left(\frac{a_t}{a_c}\right) = \tan^{-1}\left(\frac{4.00}{64.0}\right) = 3.58^{\text{o}}$$

c) $$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 = \left(1.00\ \text{rad}\right) + \frac{1}{2}\left(4.00\ \text{rad/s}^2\right)\left(2.00\ \text{s}\right) = 9.00\ \text{rad}$$

7-4- Kinetic energy of rotation

Consider a rigid body being made up of a large number of particles, with masses m_1, m_2, m_3,… at distances r_1, r_2, r_3 … respectively from the axis of rotation.
When the rigid body rotates about a fixed axis, different particles have different values of r_i and v_i but ω is the same for all. The total kinetic energy K_R of the rotating rigid object is the sum of the kinetic energies K_i of the individual particles:



$$K_R = K_1 + K_2 + K_3 + \ldots$$

$$K_R = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \ldots$$

$$K_R = \sum_i \frac{1}{2} m_i v_i^2$$

Substituting $v = \omega r$.

$$K_R = \sum_i \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

The quantity $\sum_i m_i r_i^2$ tells us how the mass of the rotating body is distributed about its axis of rotation. We call that quantity the moment of inertia (or rotational inertia) "I" of the body with respect to the axis of rotation.

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

$$K_R = \frac{1}{2} I\omega^2 \qquad \text{(Rotational kinetic energy of a rigid body)}$$

Where ω must be measured in radians per second.

7-5- Calculation of moments of inertia:

The moment of inertia of a system of discrete particles can be calculated in a straight forward way with equation:

$$I = \sum_i m_i r_i^2$$

We can evaluate the moment of inertia of a continuous rigid object by imagining the object to be divided into many small elements, each of which has mass Δm_i.

$$I = \sum_i \Delta m_i r_i^2$$

As $\Delta m_i \rightarrow 0$ $\rightarrow$ $I = \lim\limits_{\Delta m_i \to 0} \sum_i \Delta m_i r_i^2 = \int r^2 dm$

It is usually easier to calculate moments of inertia in terms of the volume of the elements rather than their mass, and we can easily make that change by using equation:

$$\rho = \frac{m}{V}$$

Where ρ is the density of the object and *V* is its volume. From this equation, the mass of a small element is: $dm = \rho dV$

$$I = \int \rho r^2 dV$$

Ex: A machine part as shown in the figure below consists of three disks linked by lightweight struts. (a) What is this body's moment of inertia about an axis through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about an axis through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about the axis through A with angular speed 4 rad/s.

Soln:

a)

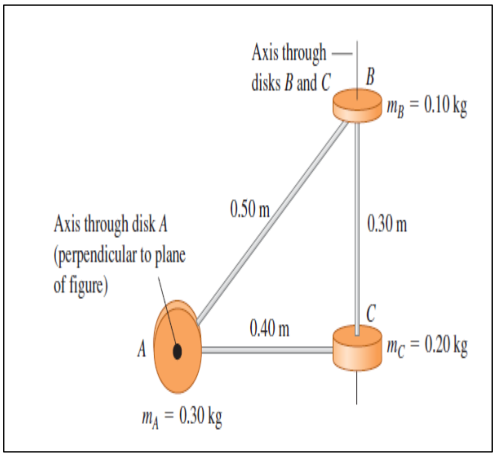
$$I_A = \sum m_i r_i^2 = (0.10\ \text{kg})(0.50\ \text{m})^2 + (0.20\ \text{kg})(0.40\ \text{m})^2 = 0.057\ \text{kg}\cdot\text{m}^2$$

b) The particles at *B* and *C* both lie on axis *BC*, so neither particle contributes to the moment of inertia. Hence only *A* contributes:

$$I_{BC} = \sum m_i r_i^2 = (0.30\ \text{kg})(0.40\ \text{m})^2 = 0.048\ \text{kg}\cdot\text{m}^2$$

c)

$$K_R = \tfrac{1}{2} I_A \omega^2 = \tfrac{1}{2}(0.057\ \text{kg}\cdot\text{m}^2)(4.0\ \text{rad/s})^2 = 0.46\ \text{J}$$



The moment of inertia about axis *A* is greater than that about axis *BC*. Hence of the two axes it's easier to make the machine part rotate about axis *BC*.

Ex: Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy-plane. We shall assume the radii of the spheres are small compared with the dimensions of the rods. (A) If the system rotates about the y-axis (Fig. a) with an angular speed ω, find the moment of inertia and the rotational kinetic energy of the system about this axis. (B) Suppose the system rotates in the xy-plane about an axis (the z-axis) through the center of the system (Fig. b). Calculate the moment of inertia and rotational kinetic energy about this axis.

Soln:

a)

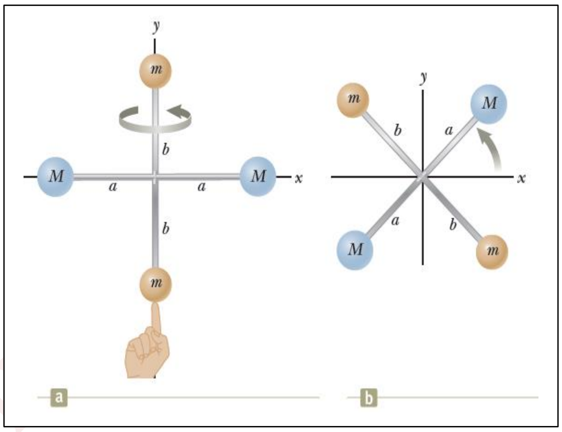
$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

$$K_R = \tfrac{1}{2} I_y \omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = \boxed{Ma^2\omega^2}$$

b)

$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

$$K_R = \tfrac{1}{2} I_z \omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = \boxed{(Ma^2 + mb^2)\omega^2}$$



Ex: The four particles in the figure below are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the xy-plane about the z-axis with an angular speed of 6 rad/s, calculate (a) the moment of inertia of the system about the z-axis and (b) the rotational kinetic energy of the system.

Soln:

a)

$$I_z = \sum_i m_i r_i^2$$

$$r_1 = r_2 = r_3 = r_4$$

$$r = \sqrt{(3.00\ \text{m})^2 + (2.00\ \text{m})^2} = \sqrt{13.0}\ \text{m}$$

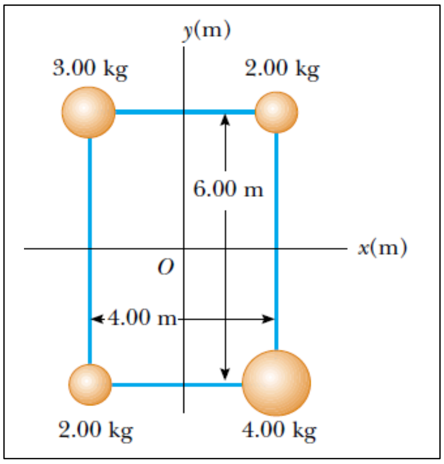
$$I = \left[\sqrt{13.0}\ \text{m}\right]^2 [3.00 + 2.00 + 2.00 + 4.00]\ \text{kg}$$

$$= \boxed{143\ \text{kg}\cdot\text{m}^2}$$

b)

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}(143\ \text{kg}\cdot\text{m}^2)(6.00\ \text{rad/s})^2$$

$$= \boxed{2.57 \times 10^3\ \text{J}}$$



Ex: Rigid rods of negligible mass lying along the y-axis connect three particles as in the figure below. If the system rotates about the x-axis with an angular speed of 2 rad/s, find (a) the moment of inertia about the x-axis and the total rotational kinetic energy, and (b) the tangential speed of each particle and the total kinetic energy

Soln:

$$m_1 = 4.00 \text{ kg},\ r_1 = |y_1| = 3.00 \text{ m};$$

$$m_2 = 2.00 \text{ kg},\ r_2 = |y_2| = 2.00 \text{ m};$$

$$m_3 = 3.00 \text{ kg},\ r_3 = |y_3| = 4.00 \text{ m};$$

$$\omega = 2.00 \text{ rad/s about the } x\text{-axis}$$

a)

$$I_x = \sum_i m_i r_i^2$$

$$I_x = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$I_x = 4.00(3.00)^2 + 2.00(2.00)^2 + 3.00(4.00)^2 = \boxed{92.0 \text{ kg}\cdot\text{m}^2}$$

$$K_R = \frac{1}{2} I_x \omega^2 = \frac{1}{2}(92.0)(2.00)^2 = \boxed{184 \text{ J}}$$

b)

$$v_1 = r_1\omega = 3.00(2.00) = \boxed{6.00 \text{ m/s}}$$

$$v_2 = r_2\omega = 2.00(2.00) = \boxed{4.00 \text{ m/s}}$$

$$v_3 = r_3\omega = 4.00(2.00) = \boxed{8.00 \text{ m/s}}$$

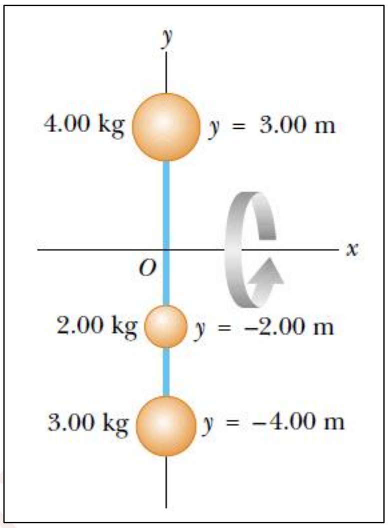
c)

$$K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2}(4.00)(6.00)^2 = 72.0 \text{ J}$$

$$K_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2}(2.00)(4.00)^2 = 16.0 \text{ J}$$

$$K_3 = \frac{1}{2} m_3 v_3^2 = \frac{1}{2}(3.00)(8.00)^2 = 96.0 \text{ J}$$

$$K = K_1 + K_2 + K_3 = 72.0 + 16.0 + 96.0 = \boxed{184 \text{ J}}$$



Ex: Uniform thin rod of length L and mass M. Calculate the moment of inertia: (A) about an axis perpendicular to the rod (the y-axis) and passing through its center of mass. (B) About an axis perpendicular to the rod (the y-axis) and passing through one end.

Soln:

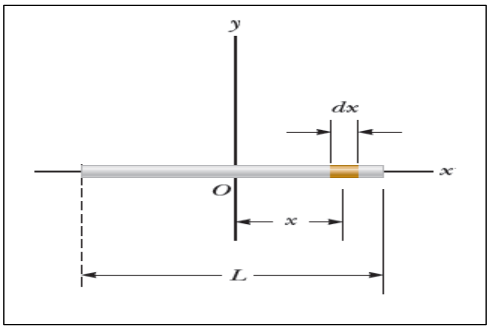
The mass per unit length (the linear mass density) can be written as $\lambda = M/L$ for uniform rod.
The rod is divided into elements of length dx, the mass of each element is $dm=\lambda dx$.

A)

$$dm = \lambda\, dx = \frac{M}{L}\, dx$$

$$I_y = \int r^2\, dm = \int_{-L/2}^{L/2} (x)^2 \frac{M}{L}\, dx = \frac{M}{L}\int_{-L/2}^{L/2} (x)^2\, dx$$

$$= \frac{M}{L}\left[\frac{(x)^3}{3}\right]_{-L/2}^{L/2} = \boxed{\tfrac{1}{12}ML^2}$$

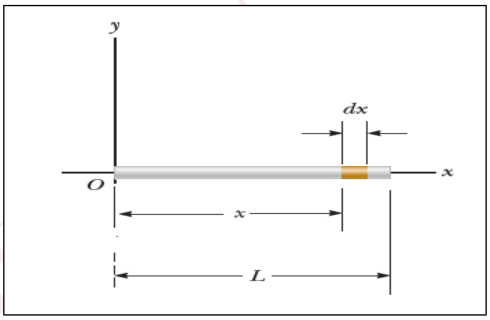


B)

$$dm = \lambda\, dx = \frac{M}{L}\, dx$$

$$I_y = \int r^2\, dm = \int_0^L x^2 \frac{M}{L}\, dx = \frac{M}{L}\int_0^L x^2\, dx$$

$$= \frac{M}{L}\left[\frac{(x)^3}{3}\right]_0^L = \boxed{\tfrac{1}{3}ML^2}$$



Ex: A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis (the *z* axis).

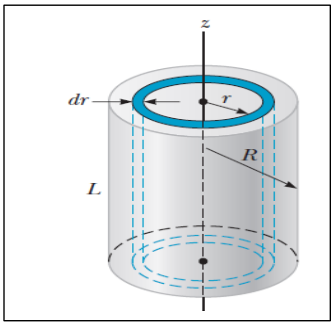
Soln:
Divide the cylinder into many cylindrical shells, each having radius r, thickness dr, and length L. The density of the cylinder is $\rho = M/V$. The volume dV of each shell is: $dV = L(2\pi r)dr$.

$$dm = \rho\, dV = \rho L(2\pi r)\, dr$$

$$I_z = \int r^2\, dm = \int r^2[\rho L(2\pi r)\, dr] = 2\pi\rho L\int_0^R r^3\, dr = \tfrac{1}{2}\pi\rho L R^4$$

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$$

$$I_z = \tfrac{1}{2}\pi\left(\frac{M}{\pi R^2 L}\right)LR^4 = \boxed{\tfrac{1}{2}MR^2}$$



Ex: A uniform hollow cylinder has inner radius R_1, outer radius R_2, mass M, and length L. Calculate its moment of inertia about its central axis (the *z* axis).

Soln:
Divide the cylinder into many cylindrical shells, each having radius r, thickness dr, and length L. The density of the cylinder is $\rho = M/V$. The volume dV of each shell is:
$dV = L(2\pi r)dr$.

$$dm = \rho\, dV = \rho L(2\pi r)\, dr$$

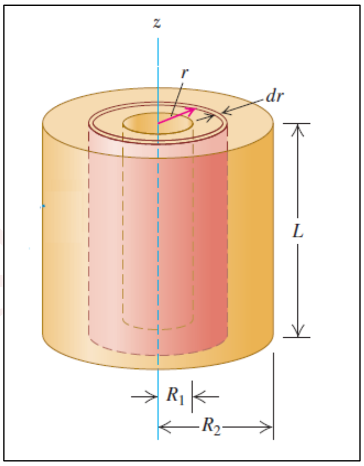
$$I_z = \int r^2\, dm = \int r^2[\rho L(2\pi r)\, dr] = 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr = \tfrac{1}{2}\pi\rho L\,(R_2^4 - R_1^4)$$

$$\rho = \frac{M}{V} = \frac{M}{\pi(R_2^2 - R_1^2)L}$$

$$I_z = \tfrac{1}{2}\pi \frac{M}{\pi(R_2^2 - R_1^2)L} L(R_2^4 - R_1^4)$$

$$I_z = \tfrac{1}{2}\pi \frac{M}{\pi(R_2^2 - R_1^2)L} L\,(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

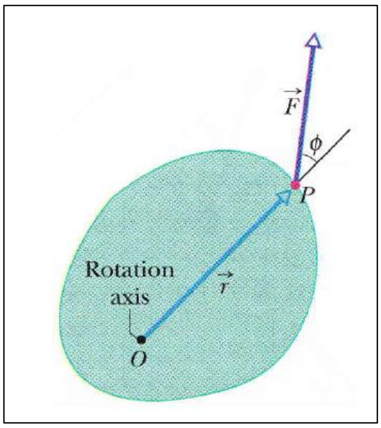
$$I_z = \tfrac{1}{2}M(R_1^2 + R_2^2)$$



7-6- Torque

When a force $\vec{F}$ is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a quantity called **torque** τ which is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{.........(1)}$$



Where r is the distance between the rotation axis and the point of application of $\vec{F}$.

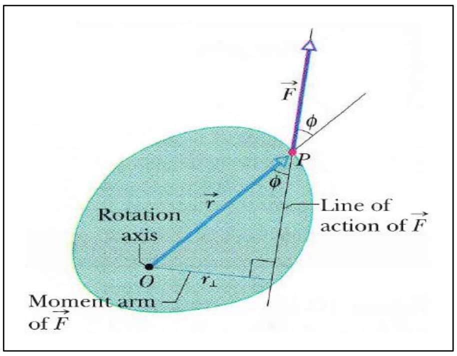
The torque is a vector quantity its magnitude is:

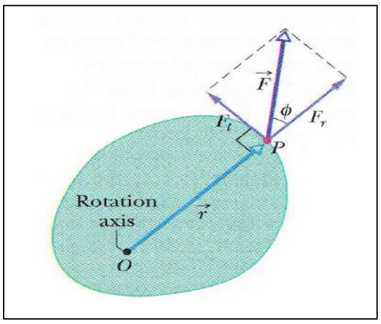
$$\tau = rF\sin\phi$$

$$\tau = (r\sin\phi)F = r_\perp F \qquad \text{........(2)}$$

$$\tau = r(F\sin\phi) = rF_t \qquad \text{........(3)}$$

Where $r_\perp$: is called moment arm (lever arm) of $\vec{F}$, and F_t is the tangential component of $\vec{F}$.
The SI unit of torque is the newton-meter (N.m)





Ex: A force of $\vec{F} = (2\hat{i} + 3\hat{j})N$ is applied to an object that is pivoted about a fixed axis aligned along the *z* coordinate axis. If the force is applied at a point located at $\vec{r} = (4\hat{i} + 5\hat{j})m$, find the torque vector.

Soln:

$\vec{\tau} = \vec{r} \times \vec{F} \quad \rightarrow \quad \vec{\tau} = (4\hat{i} + 5\hat{j}) \times (2\hat{i} + 3\hat{j}) = 2\hat{k}\ N.m$

7-7- Newton's Second law of rotation

Consider a rigid object of arbitrary shape rotating about a fixed axis passing through a point *O* as in in the figure below. The object can be regarded as a collection of particles of mass m_i.

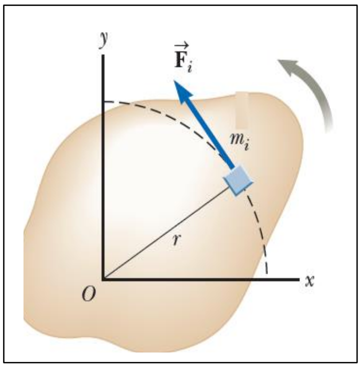
From Newton's second law an external tangential force of magnitude F_i exerted on a particle of mass m_i can accelerate it along the path with an acceleration of a_i .

$$F_i = m_i a_i$$

$$r_i F_i = m_i a_i r_i$$

$$r_i F_i = m_i (\alpha r_i) r_i = \alpha (m_i r_i^2)$$

$$\tau_i = I\alpha$$

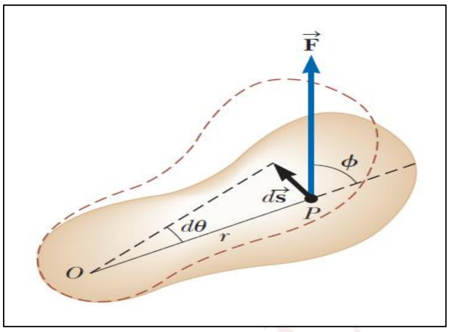


The torques on all of the particles making up the rigid object to obtain the net torque on the object about an axis through *O* due to all external forces:

$\sum_i \tau_i = I\alpha$ (Newton's second law in angular form)

7-8- Work and rotational Kinetic energy

Suppose a single external force $\vec{F}$ is applied at *P*. The work done on the object by $\vec{F}$ as its point of application rotates through an infinitesimal distance $ds = r d\theta$ is:



$$dW = \vec{F} \cdot d\vec{s} = (F \sin\phi) r d\theta$$

$$dW = \tau d\theta$$

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

$$dW = I\omega d\omega$$

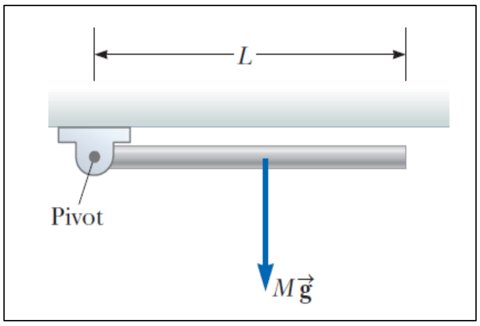
The work W done by the net external force acting on a rotating system:

$$W = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

$$W = \Delta K_R = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \qquad \text{(Work-Kinetic Energy Theorem)}$$

Ex A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane as in figure below. The rod is released from rest in the horizontal position. (A) What is the initial angular acceleration of the rod and the initial translational acceleration of its right end? (B) What is its angular speed when it reaches its lowest position?

Soln:



(A)

$$\sum \tau = rF = \frac{L}{2}Mg$$

$$\sum \tau = I\alpha \quad , \quad I = \frac{1}{3}ML^2$$

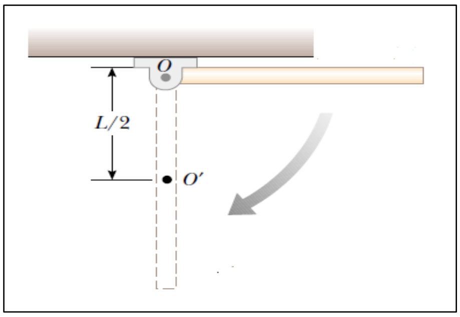
$$\frac{L}{2}Mg = I\alpha \rightarrow \frac{L}{2}Mg = \frac{1}{3}ML^2\alpha$$

$$\alpha = \frac{3g}{2L} \rightarrow a_t = r\alpha = L\frac{3g}{2L} = \frac{3g}{2}$$

(B) $E_i = E_f \rightarrow K_i + U_i = K_f + U_f$

$$0 + \frac{1}{2}MgL = \frac{1}{2}I\omega^2 + 0$$

$$\frac{1}{2}MgL = \frac{1}{2}\frac{1}{3}ML^2\omega^2 \rightarrow \omega = \sqrt{\frac{3g}{L}}$$



Ex: A uniform disk, with mass M=2.5 kg and radius R=20 cm, mounted on a fixed horizontal axle. A block with mass m=1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. (A) Find the acceleration of the falling block, (B) the angular acceleration of the disk, and (C) the tension in the cord. The cord does not slip, and there is no friction at the axle. (D) If the disk start from rest at time t=0 what is its kinetic energy at t=2.5 s.

Soln:
(A)
Apply Newton second law on the block

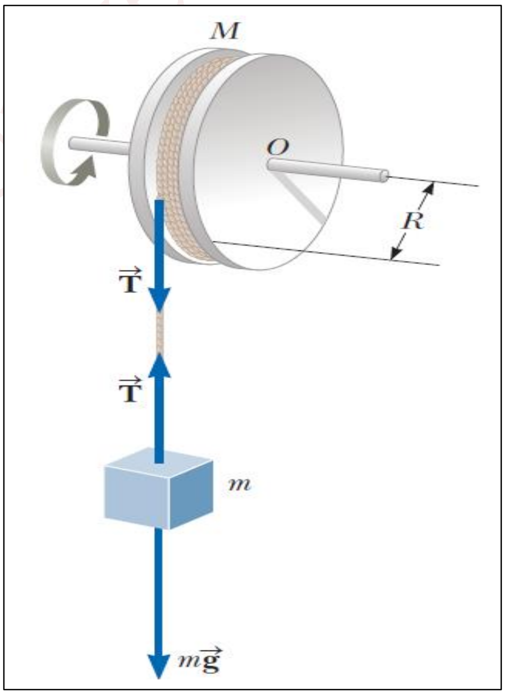
$$\sum F_y = ma_y \rightarrow T - mg = ma \rightarrow T = ma + mg \quad .....(1)$$

Apply Newton second law in rotational motion on the disk

$$\sum \tau = I\alpha \rightarrow -TR = \frac{1}{2}MR^2\alpha$$

$$-T = \frac{1}{2}MR\left(\frac{a}{R}\right) \rightarrow T = -\frac{1}{2}Ma \quad .....(2)$$

Equate eq.(1) and (2)



$$ma + mg = -\frac{1}{2}Ma \rightarrow \left(m + \frac{1}{2}M\right)a = -mg$$

$$a = -\frac{2mg}{(2m+M)} \rightarrow a = -\frac{2(1.2)(9.8)}{(2(1.2)+2.5)} = -4.8\ \text{m/s}^2$$

(B) $$T = -\frac{1}{2}Ma = -\frac{1}{2}(2.5\ \text{kg})(-4.8\ \text{m/s}^2) = 6\ \text{N}$$

(C) $$\alpha = \frac{a}{R} = \frac{-4.8\ \text{m/s}^2}{0.2\ \text{m}} = -24\ \text{rad/s}^2$$

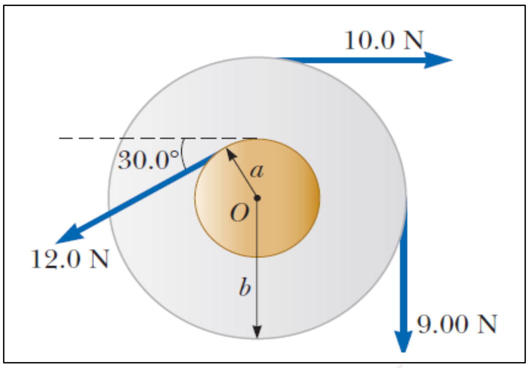
(D) $$\omega_f = \omega_i + \alpha t \rightarrow \omega_f = 0 + (-24)(2.5) = 60\ \text{rad/s}$$

$$K_R = \frac{1}{2}I\omega^2 \rightarrow K_R = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$$

$$K_R = \frac{1}{2}\left(\frac{1}{2}(2.5)(0.2)^2\right)(60)^2 = 90\ \text{J}$$

Ex: Find the net torque on the wheel in the figure below about the axle through *O*, taking a=10 cm and b=25 cm.

Soln:



$$\sum_i \tau_i = \sum_i F_i r_i$$

$$\sum_i \tau_i = (10)(0.25) + (9)(0.25) - (12)(0.10)$$

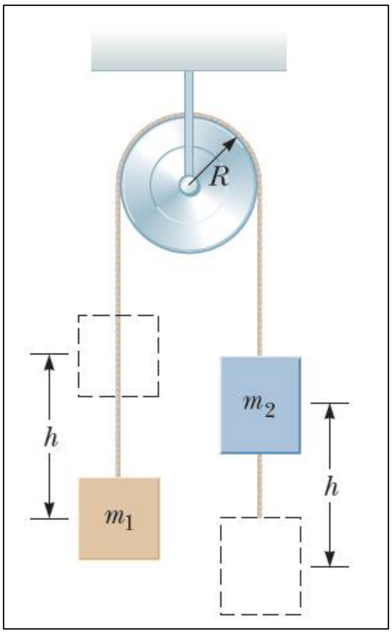
$$= 2.50 + 2.25 - 1.20 = 3.55 \text{ N.m}$$

Ex: Two blocks having different masses m_1 and m_2 are connected by a string passing over a pulley as shown in Figure below. The pulley has a radius *R* and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the translational speeds of the blocks after block 2 descends through a distance h and find the angular speed of the pulley at this time.

Soln:

$$E_i = E_f \rightarrow \quad K_i + U_i = K_f + U_f$$

$$K_{1i} + U_{1i} + K_{2i} + U_{2i} + K_{Ri} = K_{1f} + U_{1f} + K_{2f} + U_{2f} + K_{Rf}$$



$$0 + 0 + 0 + m_2 gh + 0 = \frac{1}{2} m_1 v_{1f}^2 + m_1 gh + \frac{1}{2} m_2 v_{2f}^2 + 0 + \frac{1}{2} I\omega_f^2$$

$$(m_2 - m_1) gh = \frac{1}{2}\left(m_1 v_f^2 + m_2 v_f^2 + I\frac{v_f^2}{R^2}\right)$$

$$(m_2 - m_1) gh = \frac{1}{2} v_f^2 \left(m_1 + m_2 + \frac{I}{R^2}\right)$$

$$v_f = \sqrt{\frac{2(m_2 - m_1) gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}}$$

$$\omega_f = \frac{v_f}{R} = \frac{1}{R}\sqrt{\frac{2(m_2 - m_1) gh}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}}$$