Chapter Six
Linear momentum and collisions

6-1- Linear momentum

The **linear momentum** $\vec{P}$ of a particle or an object is defined to be the product of the mass and velocity of the particle.

$$\vec{P} = m\vec{v} \qquad \text{……..(1)}$$

Linear momentum is a vector quantity because it equals the product of a scalar quantity *m* and a vector quantity $\vec{v}$. Its direction is along $\vec{v}$, it has dimensions ML/T, and its SI unit is kg.m/s.

If a particle is moving in an arbitrary direction, $\vec{P}$ has three components, and Eq. (1) is equivalent to the component equations

$$P_x = mv_x \qquad P_y = mv_y \qquad P_z = mv_z$$

We can express Newton's second law of motion in terms of momentum

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

If the mass m is constant

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{P}}{dt} \qquad \text{……(2)} \quad \text{(Newton's second law)}$$

"The net force acting on a particle equals the time rate of change of momentum of the particle".

6-2- Isolated system (Momentum)

Consider an isolated system of two particles as shown in the figure below, with masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$ at an instant of time. Because the system is isolated, the only force on one particle is that from the other article. If a force $\vec{F}_{12}$ from particle 1 acts on particle 2, there must be a second force $\vec{F}_{21}$ from particle 2 exerts on particle 1 equal in magnitude but opposite in direction. That is, form a Newton's third law:

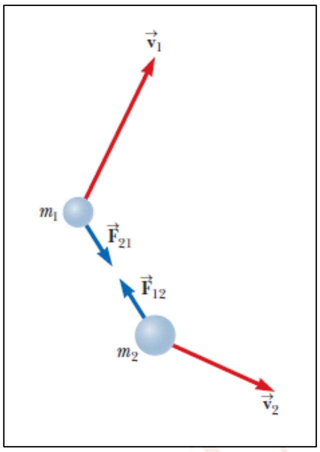
$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\vec{F}_{21} + \vec{F}_{12} = 0$$

$$m_1\vec{a}_1 + m_2\vec{a}_2 = 0$$

$$m_1\frac{d\vec{v}_1}{dt}+m_2\frac{d\vec{v}_2}{dt}=0$$

If the masses m_1 and m_2 are constant



$$\frac{d(m_1\vec{v}_1)}{dt}+\frac{d(m_2\vec{v}_2)}{dt}=0$$

$$\frac{d}{dt}(m_1\vec{v}_1+m_2\vec{v}_2)=0$$

$$\frac{d}{dt}(\vec{P}_1+\vec{P}_2)=0$$

$$\frac{d}{dt}\vec{P}_{tot}=0$$

$\vec{P}_{tot}$ =constant

$$\vec{P}_{1i}+\vec{P}_{2i}=\vec{P}_{1f}+\vec{P}_{2f} \qquad \text{.......(3)}$$

Eq.(3) can be written

$$P_{1ix}+P_{2ix}=P_{1fx}+P_{2fx} \qquad P_{1iy}+P_{2iy}=P_{1fy}+P_{2fy} \qquad P_{1iz}+P_{2iz}=P_{1fz}+P_{2fz}$$

Whenever two or more particles in an isolated system interact, the total momentum of the system does not change.

6-3- Nonisolated system (Momentum)

Consider a particle acted on by a net force $\sum\vec{F}$ during a time interval Δt from t_i to t_f.

From Newton's second law:

$$\sum\vec{F}=\frac{d\vec{P}}{dt} \qquad \text{or} \qquad \sum\vec{F}dt=d\vec{P}$$

The change in the momentum due to $\sum\vec{F}$ during the time interval Δt from t_i to t_f

$$\int_{t_i}^{t_f}\sum\vec{F}dt=\int_{\vec{P}_i}^{\vec{P}_f}d\vec{P}=\vec{P}_f-\vec{P}_i$$

$$\int_{t_i}^{t_f}\sum\vec{F}dt=\Delta\vec{P}$$

The quantity $\int_{t_i}^{t_f}\sum\vec{F}dt$ is called impulse $\vec{J}$ and is equal to change in the linear momentum.

$$\vec{J} = \int_{t_i}^{t_f} \sum \vec{F} dt \qquad \text{........(4)}$$

$$\vec{J} = \Delta \vec{P} \qquad \text{.........(5)} \qquad \text{(Impulse-momentum theorem)}$$

The change in the momentum of a particle is equal to the impulse of the net force acting on the particle.

If the net force $\sum \vec{F}$ is constant, then eq.(4) can be written

$$\vec{J} = \sum \vec{F} \Delta t \qquad \text{.......(6)}$$

In the case $\sum \vec{F}$ varies with time, we can define an average net force $\left(\sum \vec{F}\right)_{av}$

$$\vec{J} = \left(\sum \vec{F}\right)_{av} \Delta t \qquad \text{........(7)}$$

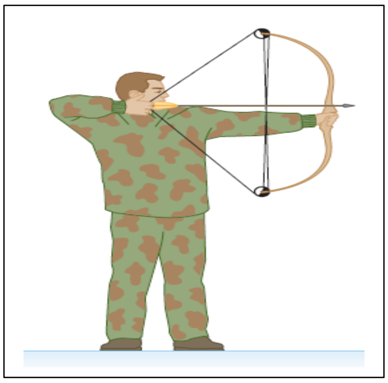
Eq.(4), can be written in component form

$$J_x = \int_{t_i}^{t_f} \sum F_x dt = P_{xf} - P_{xi} = mv_{xf} - mv_{xi} \qquad \text{........(8)}$$

$$J_y = \int_{t_i}^{t_f} \sum F_y dt = P_{yf} - P_{yi} = mv_{yf} - mv_{yi} \qquad \text{........(9)}$$

$$J_z = \int_{t_i}^{t_f} \sum F_z dt = P_{zf} - P_{zi} = mv_{zf} - mv_{zi} \qquad \text{........(10)}$$

Ex: A 60 kg archer stands at rest on frictionless ice and fires a 0.030 kg arrow horizontally at 85 m/s as shown in the figure below. With what velocity does the archer move across the ice after firing the arrow?



Soln:

$$\Delta \vec{P} = 0 \quad \rightarrow \quad \vec{P}_f - \vec{P}_i = 0 \quad \rightarrow \quad \vec{P}_i = \vec{P}_f$$

$$\left(m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}\right) = \left(m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}\right)$$

$$\left(m_1 (0) + m_2 (0)\right) = \left(m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}\right)$$

$$0 = \left(m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}\right) \quad \rightarrow \quad \vec{v}_{1f} = -\left(\frac{m_2}{m_1}\right) \vec{v}_{2f}$$

$$\vec{v}_{1f} = -\left(\frac{0.03}{60}\right)\left(85\hat{i}\right) = -0.042\hat{i} \text{ m/s}$$

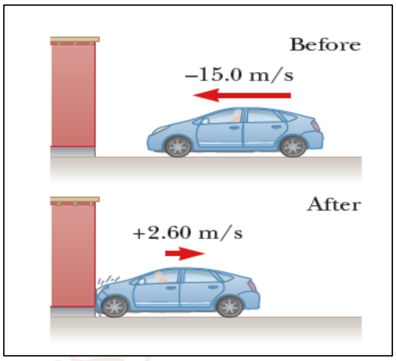
Ex: In a particular crash test, a car of mass 1500 kg collides with a wall as shown in figure below. The initial and final velocities of the car are $\vec{v}_i = -15\,\hat{i}$ m/s and $\vec{v}_f = 2.6\,\hat{i}$ m/s, respectively. If the collision lasts 0.15 s, find the impulse caused by the collision and the average net force exerted on the car.

Soln:

$$\vec{J} = \Delta\vec{P} \quad \rightarrow \quad \vec{J} = \vec{P}_f - \vec{P}_i$$

$$\vec{J} = m(\vec{v}_f - \vec{v}_i) = 1500(2.6\hat{i} + 15\hat{i}) = 2.64\times10^4\ \hat{i}\ \text{Kg.m/s}$$

$$\vec{J} = \left(\sum\vec{F}\right)_{av}\Delta t \quad \rightarrow \left(\sum\vec{F}\right)_{av} = \frac{\vec{J}}{\Delta t} = \frac{2.64\times10^4\ \hat{i}}{0.15} = 1.76\times10^5\ \hat{i}\ \text{N}$$



Ex: An estimated force-time curve for a baseball struck by a bat is shown in Figure below. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

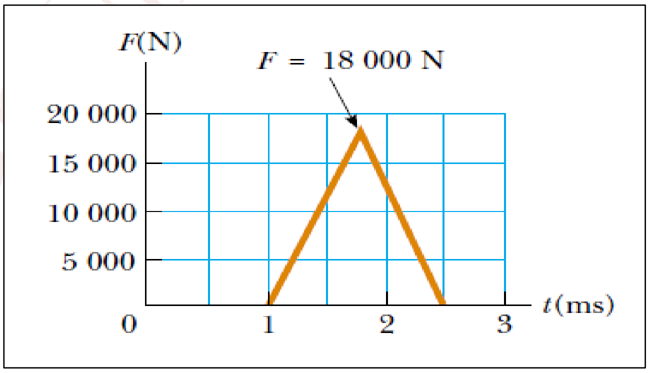
Soln:

a) $J = \int_{t_i}^{t_f} \sum F dt$ = area under curve

$$J = \frac{1}{2}(1.5\times10^{-3}\text{s})(18000\ \text{N}) = 13.5\ \text{N.s}$$

b) $F = \dfrac{13.5\ \text{N.s}}{1.5\times10^{-3}} = 9000\ \text{N}$

c) From graph, we see that F_{max}=18000 N



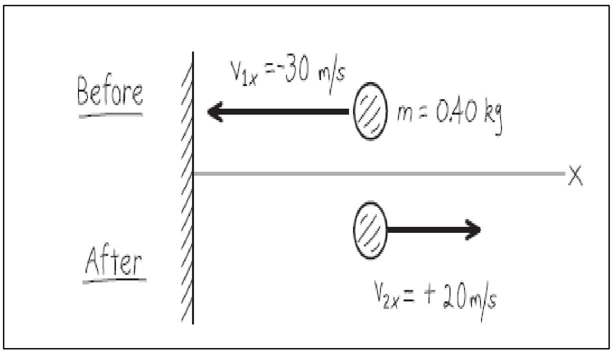
Ex: You throw a ball with a mass of 0.4 kg against a brick wall. It hits the wall moving horizontally to the left at 30 m/s and rebounds horizontally to the right at 20 m/s (a) Find the impulse of the net force on the ball during its collision with the wall. (b) If the ball is in contact with the wall for 0.010 s, find the average horizontal force that the wall exerts on the ball during the impact

Soln:

$$J_x = \Delta P_x \quad \rightarrow \quad J_x = P_{xf} - P_{xi}$$

$$J_x = m(v_{xf} - v_{xi}) = (0.4)(20+30) = 20\ \text{Kg.m/s}$$

$$J_x = \left(\sum F_x\right)_{av}\Delta t \quad \rightarrow \left(\sum F_x\right)_{av} = \frac{J_x}{\Delta t} = \frac{20}{0.01} = 2000\ \text{N}$$



6-4- Collisions in one dimension

The term collision represents an event during which two particles come close to each other and interact by means of forces.

Collisions are categorized as being either elastic or inelastic depending on whether or not kinetic energy is conserved.

An ***elastic collision*** between two objects is one in which the total kinetic energy (as well as total momentum) of the system is the same before and after the collision. For example, collision between billiard balls.

An ***inelastic collision*** is one in which the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved). Inelastic collisions are of two types. When the objects stick together after they collide, as happens when a meteorite collides with the Earth, the collision is called *perfectly inelastic.*

When the colliding objects do not stick together but some kinetic energy is transformed or transferred away, as in the case of a rubber ball colliding with a hard surface, the collision is called *inelastic.*

6-4-1- Perfectly inelastic collisions

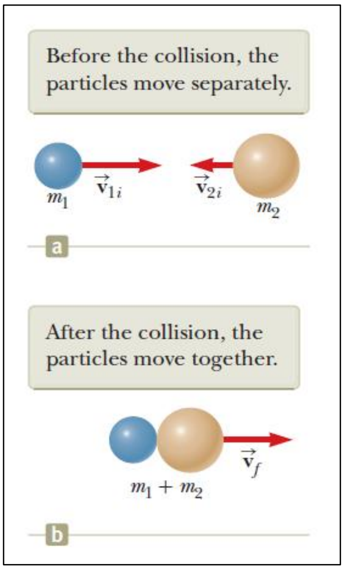
Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ along the same straight line as shown in figure below. The two particles collide head on, stick together, and then move with some common velocity $\vec{v}_f$ after the collision.

Because the momentum of an isolated system is conserved in any collision, we can say that the total momentum before the collision equals the total momentum of the composite system after the collision:

$$\Delta\vec{P} = 0 \quad \rightarrow \quad \vec{P}_i = \vec{P}_f$$

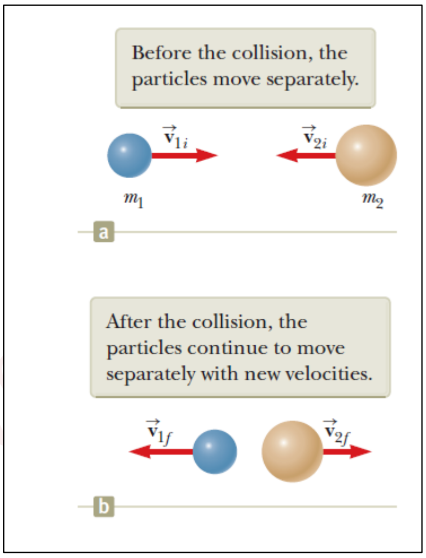
$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{(m_1 + m_2)}$$



6-4-2- Elastic collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ along the same straight line as shown in figure below. The two particles collide head on, then leave the collision site with different velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$. In an elastic collision, both the momentum and kinetic energy of the system are conserved. Therefore, considering velocities along the horizontal direction



$$P_i = P_f \quad \rightarrow \quad m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \qquad \text{……….(1)}$$

$$K_i = K_f \quad \rightarrow \quad \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad \text{………(2)}$$

Divided eq.(1) by eq.(2):

$$v_{1i} + v_{1f} = v_{2f} + v_{2i} \qquad \text{……...(3)}$$

Ex: An 1800 kg car stopped at a traffic light is struck from the rear by a 900 kg car. The two cars become entangled, moving along the same path as that of the originally moving car. If the smaller car were moving at 20 m/s before the collision, what is the velocity of the entangled cars after the collision?

Soln:

$m_1 = 1800\,\text{Kg}$, $v_{1i} = 0$ $\quad m_2 = 900\,\text{Kg}$, $v_{2i} = 20\,\text{m/s}$

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{(m_1 + m_2)} \rightarrow v_f = \frac{m_2 v_{2i}}{(m_1 + m_2)} = \frac{900 \times 20}{1800 + 900} = 6.67\,\text{m/s}$$

6-5- Collisions in two dimensions

For two-dimensional collisions, we obtain two component equations for conservation of momentum:

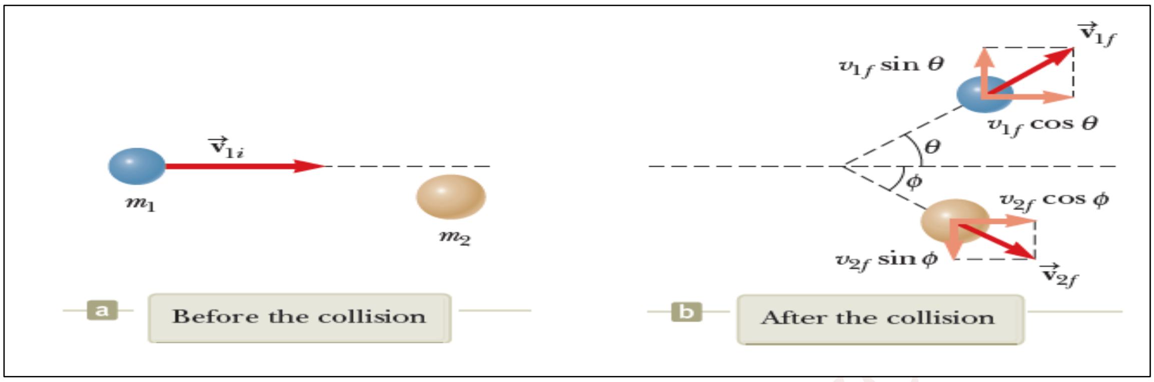
$$\Delta P_x = 0 \quad \rightarrow \quad P_{ix} = P_{fx} \quad \rightarrow \quad m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$\Delta P_y = 0 \quad \rightarrow \quad P_{iy} = P_{fy} \quad \rightarrow \quad m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Consider a specific two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2 initially at rest as in figure below. After the collision, particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle ϕ with respect to the horizontal. This event is called a *glancing* collision. Applying the law of conservation of momentum in component form and noting that the initial *y* component of the momentum of the two-particle system is zero gives:

$$m_1 v_{1i} + 0 = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\varphi$$
$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi$$

If the collision is elastic:

$$K_i = K_f \quad \rightarrow \quad \frac{1}{2} m_1 v_{1i}^2 + 0 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

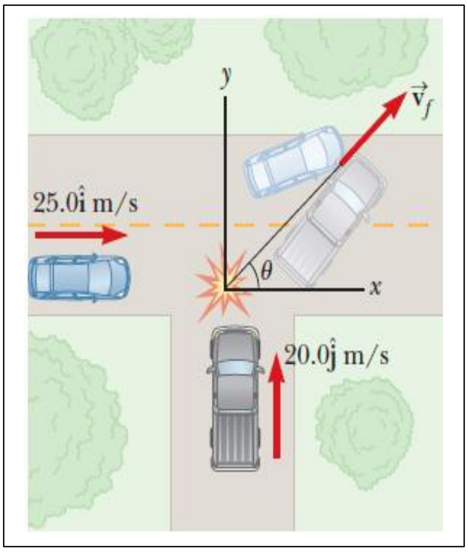


If the collision is inelastic, kinetic energy is not conserved and the above equation does *not* apply.

Ex: A 1500 kg car traveling east with a speed of 25 m/s collides at an intersection with a 2500 kg truck traveling north at a speed of 20 m/s as shown in figure below. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming the vehicles stick together after the collision.
Soln:

$$\Delta P_x = 0 \quad \rightarrow \quad \sum P_{xi} = \sum P_{xf} \quad \rightarrow \quad m_1 v_{1i} = (m_1 + m_2) v_f \cos\theta \quad \ldots(1)$$

$$\Delta P_y = 0 \quad \rightarrow \quad \sum P_{yi} = \sum P_{yf} \quad \rightarrow \quad m_2 v_{2i} = (m_1 + m_2) v_f \sin\theta \quad \ldots(2)$$



Divided eq(2) on (1)

$$\frac{m_2 v_{2i}}{m_1 v_{1i}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{m_2 v_{2i}}{m_1 v_{1i}}\right) = \tan^{-1}\left(\frac{(2500\ \mathrm{Kg})(20\ \mathrm{m/s})}{(1500\ \mathrm{Kg})(25\ \mathrm{m/s})}\right) = 53.1^{\circ}$$

$$v_f = \frac{m_2 v_{2i}}{(m_1 + m_2)\sin\theta} = \frac{(2500\ \mathrm{Kg})(20\ \mathrm{m/s})}{(1500\ \mathrm{Kg} + 2500\ \mathrm{Kg})\sin 53.1^{\circ}} = 15.6\ \mathrm{m/s}$$

Ex: A 3 kg steel ball strikes a wall with a speed of 10 m/s at an angle of 60° with the surface. It bounces off with the same speed and angle as shown in the figure below. If the ball is in contact with the wall for 0.2 s, what is the average force exerted by the wall on the ball?

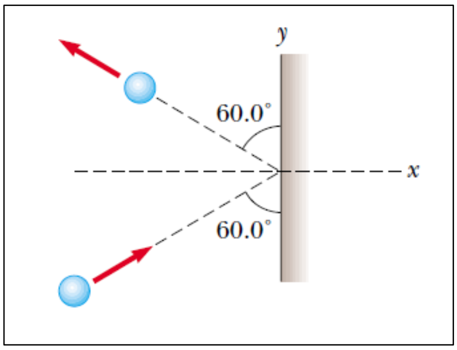
Soln:

$$\Delta P_y = m\left(v_{fy} - v_{iy}\right) = m\left(v\cos 60^o - v\cos 60^o\right) = 0$$

$$\Delta P_x = m\left(v_{fx} - v_{ix}\right) = m\left(-v\sin 60^o - v\sin 60^o\right) = -2mv\sin 60^o$$

$$\Delta P_x = -2\left(3\ \text{kg}\right)\left(10\,\text{m/s}\right)\left(0.866\right) = -52\ \text{kg.m/s}$$

$$\Delta\vec{P} = \left(\sum\vec{F}\right)_{av}\Delta t \rightarrow \left(\sum F\right)_{av} = \frac{\Delta P_x}{\Delta t} = \frac{-52}{0.2} = -260\ \text{N}$$



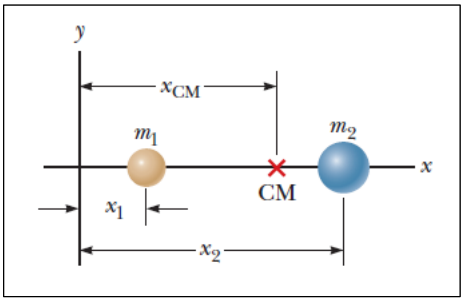
6-6- The center of mass

The center of mass of a system of particles is the point that moves as though: (1-) all of the system's mass were concentrated there, and (2-) all external forces were applied there.

6-6-1- System of particles:

The center of mass of the pair of particles described in figure below is located on the *x* axis and lies somewhere between the particles. Its *x* coordinate is given by

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$



The *x* coordinate of the center of mass of *n* particles is defined to be

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i} = \frac{\sum_i m_ix_i}{M} = \frac{1}{M}\sum_i m_ix_i \qquad \text{.....(1)}$$

The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{CM} = \frac{1}{M}\sum_i m_iy_i \qquad \text{......(2)}$$

$$z_{CM} = \frac{1}{M}\sum_i m_iz_i \qquad \text{.......(3)}$$

The center of mass can be located in three dimensions by its position vector $\vec{r}_{CM}$

$$\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k} \qquad \text{........(4)}$$

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i x_i \hat{i} + \frac{1}{M}\sum_i m_i y_i \hat{j} + \frac{1}{M}\sum_i m_i z_i \hat{k} = \frac{1}{M}\sum_i m_i \vec{r}_i \qquad \text{......(5)}$$

Where $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$

The velocity of the center of mass of a system of particles

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{v}_i \qquad \text{......(6)}$$

The acceleration of the center of mass of system of particles

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{a}_i \qquad \text{......(7)}$$

Newton’s second law:

$$M\vec{a}_{CM} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i \qquad \text{......(8)}$$

The total momentum of a system of particles:

$$M\vec{v}_{CM} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{P}_{tot} \qquad \text{......(9)}$$

6-6-2- Solid objects:

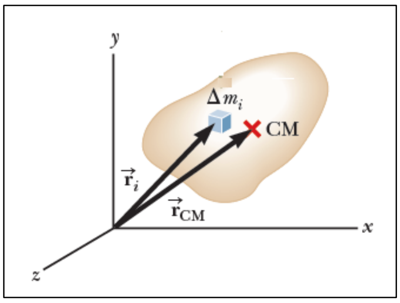
An ordinary object, contains so many particles can best treat it as a continuous distribution of matter. The particles then become differential mass elements dm, the sums of Eq.1 become integrals, and the coordinates of the center of mass are defined as:

$$x_{CM} = \frac{1}{M}\int x\,dm \qquad \text{......(10)}$$

$$y_{CM} = \frac{1}{M}\int y\,dm \qquad \text{......(11)}$$

$$z_{CM} = \frac{1}{M}\int z\,dm \qquad \text{.......(12)}$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm \qquad \text{.......(13)}$$



Ex: A system consists of three particles located as shown in figure below. Find the center of mass of the system. The masses of the particles are $m_1=m_2=1.0$ kg and $m_3=2.0$ kg.

Soln:

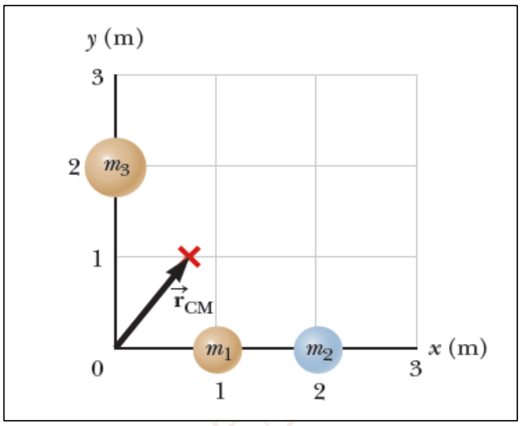
$$x_{CM}=\frac{1}{M}\sum_{i=1}^{3}m_ix_i=\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$$

$$=\frac{(1\text{ kg})(1\text{ m})+(1\text{ kg})(2\text{ m})+(2\text{ kg})(0)}{(4\text{ kg})}=0.75\text{ m}$$

$$y_{CM}=\frac{1}{M}\sum_{i=1}^{3}m_iy_i=\frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}$$

$$=\frac{(1\text{ kg})(0)+(1\text{ kg})(0)+(2\text{ kg})(2\text{ m})}{(4\text{ kg})}=1\text{ m}$$

$$\vec{r}_{CM}=x_{CM}\hat{i}+y_{CM}\hat{j}=\left(0.75\hat{i}+\hat{j}\right)\text{ m}$$



Ex: (A) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length. (B) Suppose a rod is nonuniform such that its mass per unit length varies linearly with x according to the expression $\lambda=\alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Soln:

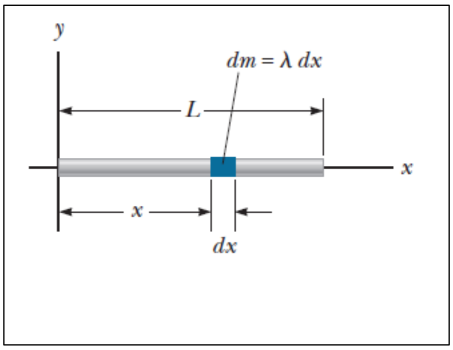
The mass per unit length (the linear mass density) can be written as $\lambda=M/L$ for the uniform rod. If the rod is divided into elements of length dx, the mass of each element is $dm=\lambda dx$.

(A) $$x_{CM}=\frac{1}{M}\int x\,dm=\frac{1}{M}\int_0^L x\lambda dx=\frac{\lambda}{M}\frac{x^2}{2}\Big|_0^L=\frac{\lambda L^2}{2M}$$

Substitute $\lambda=M/L$

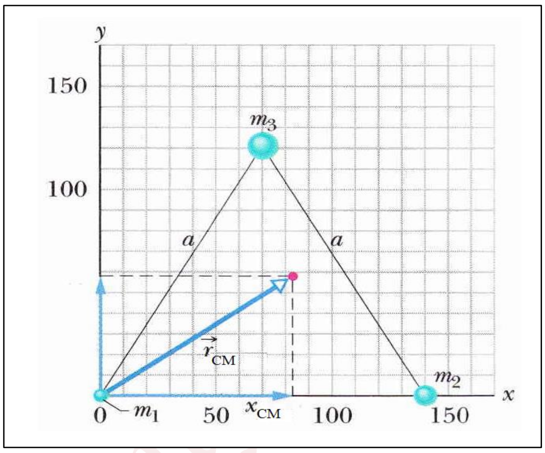
$$x_{CM}=\frac{L^2}{2M}\left(\frac{M}{L}\right)=\frac{1}{2}L$$

(B) $$x_{CM}=\frac{1}{M}\int x\,dm=\frac{1}{M}\int_0^L x\lambda dx=\frac{\alpha}{M}\int_0^L x^2dx==\frac{\alpha}{M}\frac{x^3}{3}\Big|_0^L=\frac{\alpha L^3}{3M}$$



Ex: Three particles of masses m_1= 1.2 kg, m_2= 2.5 kg, and m_3= 3.4 kg form an equilateral triangle of edge length a=140 cm. Where is the center of mass of this system?

Soln:



$$x_{CM} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(140\text{ cm}) + (3.4\text{ kg})(70\text{ cm})}{(7.1\text{ kg})} = 83\text{ cm}$$

$$y_{CM} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.2\text{k g})(0) + (2.5\text{ kg})(0\text{ cm}) + (3.4\text{ kg})(120\text{ cm})}{(7.1\text{ kg})} = 58\text{ cm}$$

$$\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} = \left(83\hat{i} + 58\hat{j}\right)\text{ cm}$$

Ex: You have been asked to hang a metal sign from a single vertical string. The sign has the triangular shape shown in figure below. The bottom of the sign is to be parallel to the ground. At what distance from the left end of the sign should you attach the support string?

Soln:

We assume the triangular sign has a uniform density and total mass M.

ρ =density of the metal, t= thickness of the metal sign

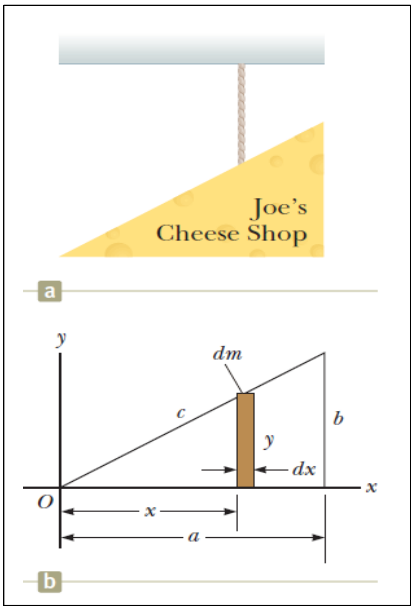
$$\rightarrow \quad dm = \rho y t dx \quad , \quad \rho = \frac{M}{V} = \frac{dm}{ytdx}$$

$V = \frac{1}{2}abt$ = volume of the triangle

$$dm = \rho y t\, dx = \left(\frac{M}{\frac{1}{2}abt}\right) yt\, dx = \frac{2My}{ab}dx$$

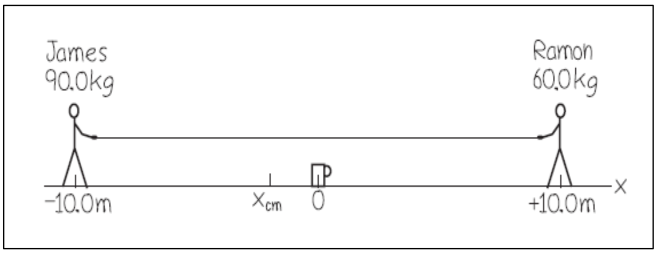
$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^a x\frac{2My}{ab}dx = \frac{2}{ab}\int_0^a xy\, dx$$

$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right)dx = \frac{2}{a^2}\int_0^a x^2\, dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a = \tfrac{2}{3}a$$



Ex: James (mass 90.0 kg) and Ramon (mass 60.0 kg) are 20.0 m apart on a frozen pond. Midway between them is a mug. They pull on the ends of a light rope stretched between them. When James has moved 6.0 m toward the mug, how far and in what direction has Ramon moved?

Soln:



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \rightarrow \quad x_{CM} = \frac{(90\text{ kg})(-10\text{ m}) + (60\text{ kg})(10\text{ m})}{(90\text{ kg} + 60\text{ kg})} = -2\text{ m}$$

When James moves 6.0 m toward the mug, his new *x*-coordinate is -4 m. The center of mass doesn't move, so

$$-2\text{ m} = \frac{(90\text{ kg})(-4\text{ m}) + (60\text{ kg})(x_2)}{(90\text{ kg} + 60\text{ kg})} \quad \rightarrow \quad x_2 = 1\text{ m}$$