⑥

* To eliminate a'_{12}: Pivot Row is R_2, and pivot element is a'_{22}.

New $R_1 = R_1 - R_2\left(\frac{a'_{12}}{a'_{22}}\right)$, we get:

$$\begin{bmatrix} a''_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

* New the set of eq.s is:

$$\begin{bmatrix} a''_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} C''_1 \\ C'_2 \\ C_3 \end{bmatrix}$$

or

$$a''_{11} X_1 = C''_1$$
$$a'_{21} X_1 + a'_{22} X_2 = C'_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3$$

we can solve for X_1, X_2 and X_3 by forward substitution

B- Backward substitution :-

$$\left.\begin{aligned} a_{11} X_1 + a_{12} X_2 + a_{13} X_3 &= C_1 \\ a_{21} X_1 + a_{22} X_2 + a_{23} X_3 &= C_2 \\ a_{31} X_1 + a_{32} X_2 + a_{33} X_3 &= C_3 \end{aligned}\right\} \Rightarrow \begin{aligned} a_{11} X_1 + a_{12} X_2 + a_{13} X_3 &= C_1 \text{ --- ①} \\ a'_{22} X_2 + a'_{23} X_3 &= C'_2 \text{ --- ②} \\ a''_{33} X_3 &= C''_3 \text{ --- ③} \end{aligned}$$

⑦

From ③ $\Rightarrow$ find (x_3).

From ② $\Rightarrow$ find (x_2) (using (x_3))

From ① $\Rightarrow$ find (x_1) (using (x_3) and (x_2))

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

- Elimination procedure

* Augmented matrix:

$$\begin{array}{c} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{21} and a_{31}: pivot row is (R_1) and pivot element is (a_{11}):

* a_{21} elimination:

$$\text{New } R_2 = R_2 - R_1 \left(\frac{a_{21}}{a_{11}}\right)$$

* a_{31} elimination:

$$\text{New } R_3 = R_3 - R_1 \left(\frac{a_{31}}{a_{11}}\right)$$

* we get

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a'_{22} & a'_{23} & c'_2 \\ 0 & a'_{32} & a'_{33} & c'_3 \end{array}\right]$$

* To eliminate a_{32}: Pivot row is R_2, and pivot element is a_{22}:

New $R_3 = R_3 - R_2\left(\frac{a_{32}}{a_{22}}\right)$, we get:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a'_{22} & a'_{23} & c'_2 \\ 0 & 0 & a''_{33} & c''_3 \end{array}\right]$$

* Now, the equations system is:

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = C_1$$
$$a'_{22}X_2 + a'_{23}X_3 = C'_2$$
$$a''_{33}X_3 = C''_3$$

* We can solve for X_3, X_2 and X_1 by backward substitution.

Example 1

Use backward Gaussian elimination to solve the following system of linear equations:

$$100X_1 + 80X_2 - 40X_3 = 8$$
$$200X_1 - 40X_2 + 20X_3 = 6$$
$$300X_1 + 340X_2 - 100X_3 = -6$$

(9)

Sol.

Augmented matrix is :

$$\left[\begin{array}{ccc:c} 100 & 80 & -40 & 8 \\ 200 & -40 & 20 & 6 \\ 300 & 340 & -100 & -6 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_2 - R_1\left(\frac{200}{100}\right) \\ R_3 = R_3 - R_1\left(\frac{300}{100}\right) \end{array}$$

$$\left[\begin{array}{ccc:c} 100 & 80 & -40 & 8 \\ 0 & -200 & 100 & -10 \\ 0 & 100 & 20 & -30 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_3 - R_2\left(\frac{100}{-200}\right) \end{array}$$

$$\left[\begin{array}{ccc:c} 100 & 80 & -40 & 8 \\ 0 & -200 & 100 & -10 \\ 0 & 0 & 70 & -35 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

The equations system is :

$$100X_1 + 80X_2 - 40X_3 = 8 \quad \text{---} \; (1)$$
$$-200X_2 + 100X_3 = -10 \quad \text{---} \; (2)$$
$$70X_3 = -35 \quad \text{---} \; (3)$$

* Using backward substitution :

From (3) : $X_3 = -35/70 \Rightarrow \boxed{X_3 = -0.5}$

From (2) : $-200X_2 = -10 - 100X_3$

$$X_2 = \frac{-10 - 100X_3}{-200} \Rightarrow X_2 = \frac{10 + 100(-0.5)}{200}$$

$$\boxed{X_2 = -0.2}$$

From ①: $100X_1 = 8 - 80X_2 + 40X_3$

$$X_1 = \frac{8 - 80(-0.2) + 40(-0.5)}{100}$$

$$\Rightarrow \boxed{X_1 = 0.04}$$

* The solution of the equations system are:

$X_1 = 0.04$, $X_2 = -0.2$, $X_3 = -0.5$

Exercise:

Use backward Gaussian elimination to solve the following system of linear equations:

$$3X_1 - X_2 + 2X_3 = 12$$
$$X_1 + 2X_2 + 3X_3 = 11$$
$$2X_1 - 2X_2 - X_3 = 2$$