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Chapter 3

Solving systems of linear equations

A system of linear equations is a collection of two or more linear equations involving the same set of variables (unknowns)

* In a system of linear equations, we have:

No. of eqs. = No. of unknows.

The simplest kind of linear system involves two equations and two variables, For example:

$$2x + 3y = 6$$
$$4x + 9y = 15.$$

A system of three linear equations, for example:

$$6x + 4y - 2z = 20$$
$$x - 10y - 7z = 15$$
$$-x + 30y + z = -1$$

The general form of system of linear equations defined by:-

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

②

Some properties of matrices :-

$$AX = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A is called coefficient matrix.
X is called unknowns vector.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is called square matrix 3×3.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$ is called upper triangular matrix 3×3

$$A_2 = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is called lower triangular matrix 3×3

$$A_3 = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$ is called diagonal matrix 3×3

$$A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is called identity matrix and denoted by I, 3×3.

$$A * A^{-1} = I.$$

There are two types of method to solve a system of linear equations :-

A- Direct methods.

B- Iterative methods.

A- Direct methods :-

1- Gaussian elimination :-

a- Forward Substitution :-

$$\left.\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 &= C_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 &= C_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 &= C_3 \end{aligned}\right\} \Longrightarrow \begin{aligned} a''_{11}X_1 &= C''_1 \quad \cdots (1) \\ a'_{21}X_1 + a'_{22}X_2 &= C_2 \cdots (2) \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 &= C_3 \cdots (3) \end{aligned}$$

④

Forward Substitution:

From ① $\Longrightarrow$ Find (x_1).

From ② $\Longrightarrow$ Find (x_2) {by using (x_1) from the previous step}.

From ③ $\Longrightarrow$ Find (x_3) {by using (x_1) and (x_2) from the previous steps}.

Or, we convert the coefficient matrix into a triangular form (leaving the lower elements)

U-elements

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

L-elements

Lower triangular matrix

Elimination produce

$$* \quad a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = C_1$$

$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = C_2$$

$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = C_3$$

(5)

* we write the augmented matrix:

$$\begin{array}{l l} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] & \begin{array}{ll} \text{Row 1} & (R_1) \\ \text{Row 2} & (R_2) \\ \text{Row 3} & (R_3) \end{array} \end{array}$$

* To eliminate a_{13}: pivot row is (R_3) and pivot element is a_{33}

New $R_1 = R_1 - R_3\left(\frac{a_{13}}{a_{33}}\right)$ we get:

$$\begin{bmatrix} a'_{11} & a'_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

* To eliminate a_{23}: pivot row is R_3 and pivot element is a_{33},

New $R_2 = R_2 - R_3\left(\frac{a_{23}}{a_{33}}\right)$, we get:

$$\begin{array}{l l} \left[\begin{array}{ccc|c} a'_{11} & a'_{12} & 0 & c'_1 \\ a'_{21} & a'_{22} & 0 & c'_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

⑥

* To eliminate a'_{12}: Pivot Row is R_2, and pivot element is a'_{22}.

New $R_1 = R_1 - R_2\left(\frac{a'_{12}}{a'_{22}}\right)$, we get:

$$\begin{array}{c} \begin{bmatrix} a''_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

* New the set of eq.s is:

$$\begin{bmatrix} a''_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} C''_1 \\ C'_2 \\ C_3 \end{bmatrix}$$

or

$$a''_{11} X_1 = C''_1$$

$$a'_{21} X_1 + a'_{22} X_2 = C'_2$$

$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3$$

we can solve for X_1, X_2 and X_3 by forward substitution

B- Backward substitution :-

$$\left.\begin{array}{l} a_{11} X_1 + a_{12} X_2 + a_{13} X_{3} = C_1 \\ a_{21} X_1 + a_{22} X_2 + a_{23} X_3 = C_2 \\ a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3 \end{array}\right\} \Rightarrow \begin{array}{r} a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = C_1 \text{ --- } ① \\ a'_{22} X_2 + a'_{23} X_3 = C'_2 \text{ --- } ② \\ a''_{33} X_3 = C''_3 \text{ --- } ③ \end{array}$$