## Chapter 5: Energy of a system

## 5-1- Work Done by a Constant Force

The work done on a system by a constant force $\overrightarrow{\mathrm{F}}$ is defined to be the product of the magnitude of the force, magnitude of the displacement of the point of application of the force, and the cosine of the angle between the force $\stackrel{\rightharpoonup}{\mathrm{F}}$ and the displacement vectors $\Delta \overrightarrow{\mathrm{r}}$.
$\mathrm{W}=\mathrm{F} \Delta \mathrm{r} \cos \theta=\overrightarrow{\mathrm{F}} \cdot \Delta \overrightarrow{\mathrm{r}}$
Work is a scalar quantity and its SI unit is the $\mathrm{N} \cdot \mathrm{m}$. One $\mathrm{N} \cdot \mathrm{m}=1$ joule (J).
Work is energy transferred to or from an object by means of a force acting on the object.
If W is positive, energy is transferred to the system and we say the work is done on the system.

If W is negative, energy is transferred from the system and we say the work is done by the system.

There is no work $(\mathrm{W}=0)$ if the force is perpendicular
 to the direction of the displacement.


Ex: A man cleaning a floor pulls a vacuum cleaner with a force of magnitude 50 N at an angle of $30^{\circ}$ with the horizontal. Calculate the work done by the force on the vacuum cleaner as the vacuum cleaner is displaced 3 m to the right.

Soln.
$\mathrm{W}=\mathrm{F} \Delta \mathrm{r} \cos \theta=(50)(3) \cos \left(30^{\circ}\right)=130 \mathrm{~J}$
Ex: A particle moving in the xy-plane undergoes a displacement given by $\Delta \overrightarrow{\mathrm{r}}=(2 \hat{\mathrm{i}}+3 \hat{\mathrm{j}}) \mathrm{m}$ as a constant force $\overrightarrow{\mathrm{F}}=(5 \hat{\mathrm{i}}+2 \hat{\mathrm{j}}) \mathrm{N}$ acts on the particle. Calculate the work done by $\overrightarrow{\mathrm{F}}$ on the particle.

Soln: $\quad W=\overrightarrow{\mathrm{F}} \cdot \Delta \overrightarrow{\mathrm{r}}=(5 \hat{\mathrm{i}}+2 \hat{\mathrm{j}}) \cdot(2 \hat{\mathrm{i}}+3 \hat{\mathrm{j}})=(5 \hat{\mathrm{i}} \cdot 2 \hat{\mathrm{i}})+(5 \hat{\mathrm{i}} \cdot 3 \hat{\mathrm{j}})+(2 \hat{\mathrm{j}} \cdot 2 \hat{\mathrm{i}})+(2 \hat{\mathrm{j}} \cdot 3 \hat{\mathrm{j}})=10+0+0+6=16 \mathrm{~J}$

Ex: A block of mass 2.5 kg is pushed a distance 2.2 m along a frictionless horizontal table by a constant applied force of magnitude 16 N directed at an angle $25^{\circ}$ above the horizontal as shown in figure below. Determine the work done on the block by (a) the applied force, (b) the normal force exerted by the table, (c) the gravitational force, and (d) the net force on the block.


Soln:
a) By the applied force $\rightarrow \mathrm{W}_{\text {app }}=\mathrm{Fd} \cos \theta=(16 \mathrm{~N})(2.2 \mathrm{~m}) \cos \left(25^{\circ}\right)=31.9 \mathrm{~J}$
b) By the normal force $\rightarrow \mathrm{W}_{\mathrm{n}}=\mathrm{nd} \cos \theta=(\mathrm{F} \sin \theta+\mathrm{mg})(\mathrm{d}) \cos \left(90^{\circ}\right)=0$
c) By the gravitational force $\rightarrow \mathrm{W}_{\mathrm{g}}=\mathrm{F}_{\mathrm{g}} \mathrm{d} \cos \theta=(\mathrm{mg})(\mathrm{d}) \cos \left(90^{\circ}\right)=0$
d) Net work done on the block $\rightarrow \mathrm{W}_{\text {net }}=\mathrm{W}_{\text {app }}+\mathrm{W}_{\mathrm{n}}+\mathrm{W}_{\mathrm{g}}=31.9 \mathrm{~J}$

## 5-2- Work done by a varying force

Consider a particle being displaced along the x-axis under the action of a force that varies with position. The particle is displaced in the direction of increasing x from $x=x_{i}$ to $x=x_{f}$. In such a situation, we cannot use $\mathrm{W}=\mathrm{F} \Delta \mathrm{r} \cos \theta=\overrightarrow{\mathrm{F}} \cdot \Delta \overrightarrow{\mathrm{r}}$ to calculate the work done by the force because this relationship applies only when $\overrightarrow{\mathrm{F}}$ is constant in magnitude and direction.

To find the work done by this force, we divide the total displacement into small segments of width $\Delta x$, the work done by the force during segment $\Delta x$ is approximately
$\mathrm{W} \approx \mathrm{F}_{\mathrm{x}} \Delta \mathrm{x}$
The work done by the force in the total displacement from $x=x_{i}$ to $x=x_{f}$ is
$\mathrm{W} \approx \sum_{\mathrm{x}_{\mathrm{i}}}^{\mathrm{x}_{\mathrm{f}}} \mathrm{F}_{\mathrm{x}} \Delta \mathrm{x}$
In the limit that the number of segments becomes very large and the width of each becomes very small $\lim _{\Delta x \rightarrow 0} \sum_{x_{i}}^{x_{f}} F_{x} \Delta x=\int_{x_{i}}^{x_{f}} F_{x} d x$

$W=\int_{x_{i}}^{x_{x}} F_{x} d x$
If more than one force acts on a system, the total work (net work) done on the system is just the work done by the net force

$$
\mathrm{W}_{\text {net }}=\int_{x_{i}}^{x_{f}}\left(\sum \mathrm{~F}_{\mathrm{x}}\right) \mathrm{dx}
$$

Ex: A force acting on a particle varies with $x$ as shown in figure below. Calculate the work done by the force on the particle as it moves from $x=0$ to $x=6 \mathrm{~m}$.
Soln:

$$
\begin{aligned}
& W_{\text {(Ato © (B) }}=(5.0 \mathrm{~N})(4.0 \mathrm{~m})=20 \mathrm{~J} \\
& W_{\text {(B) to © }}=\frac{1}{2}(5.0 \mathrm{~N})(2.0 \mathrm{~m})=5.0 \mathrm{~J} \\
& W_{(A) \mathrm{to} \text { © }}=W_{\text {(A) } \mathrm{to} \text { (B) }}+W_{\text {(8) to © }}=20 \mathrm{~J}+5.0 \mathrm{~J}=25 \mathrm{~J}
\end{aligned}
$$



## 5-3- Work done by a spring

A model of a common physical system for which the force varies with position is shown in figure below. If the spring is either stretched or compressed a small distance from its unstretched (equilibrium) configuration, it exerts on the block a force that can be expressed as
$\mathrm{F}_{\mathrm{s}}=-\mathrm{kx} \quad$ (Hooke's law)
where $x$ is the position of the block relative to its equilibrium ( $x=0$ ) position and $k$ is a positive constant called the force constant or the spring constant of the spring. This force law for springs is known as Hooke's law. The value of $k$ is a measure of the stiffness of the spring. Stiff springs have large $k$ values, and soft springs have small $k$ values. As can be seen from equation (3), the units of $k$ are $N / m$. Because the spring force always acts toward the equilibrium position ( $x=0$ ), it is sometimes called a restoring force.
The work $\mathrm{W}_{\mathrm{s}}$ done on the object by the spring force when the object is moved from an initial position $x_{i}$ to a final position $x_{f}$ is:

$$
\begin{equation*}
\mathrm{W}_{\mathrm{s}}=\int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} F_{\mathrm{s}} \mathrm{dx}=\int_{x_{\mathrm{i}}}^{x_{f}}(-\mathrm{kx}) \mathrm{dx}=\frac{1}{2} \mathrm{kx}_{\mathrm{i}}^{2}-\frac{1}{2} \mathrm{kx}_{\mathrm{f}}^{2} \tag{4}
\end{equation*}
$$

The work $W_{s}$ done by the spring force on the block as the block moves from $x_{i}=-X_{\max }$ to $X_{f}=0$ is: $W_{s}=\int_{x_{i}}^{x_{f}} F_{\mathrm{s}} \mathrm{dx}=\int_{-\mathrm{x}_{\text {max }}}^{0}(-\mathrm{kx}) \mathrm{dx}=\frac{1}{2} \mathrm{kx}_{\text {max }}^{2}$


Ex: A spring is hung vertically, and an object of mass $m$ is attached to its lower end. Under the action of the load mg , the spring stretches a distance $d$ from its equilibrium position.
(A) If a spring is stretched 2 cm by a suspended object having a mass of 0.55 kg , what is the force constant of the spring?
(B) How much work is done by the spring on the object as it stretches through this distance?
(C) Evaluate the work done by the gravitational force on the object.

Soln:
(A) $\sum \mathrm{F}_{\mathrm{y}}=0 \rightarrow \mathrm{~F}_{\mathrm{s}}-\mathrm{mg}=0$
$\mathrm{k}=\frac{\mathrm{mg}}{\mathrm{x}}=\frac{\mathrm{mg}}{\mathrm{d}}=\frac{0.55 \times 9.8}{2 \times 10^{-2}}=2.7 \times 10^{2} \mathrm{~N} / \mathrm{m}$

(B) $\mathrm{W}_{\mathrm{s}}=\frac{1}{2} \mathrm{kx}_{\mathrm{i}}^{2}-\frac{1}{2} \mathrm{kx}_{\mathrm{f}}^{2}=0-\frac{1}{2} \mathrm{kd}^{2}$

$$
=-\frac{1}{2}\left(2.7 \times 10^{2}\right)\left(2 \times 10^{-2}\right)=-5.4 \times 10^{-2} \mathrm{~J}
$$

(C) $\mathrm{W}=\overrightarrow{\mathrm{F}} \cdot \Delta \stackrel{\rightharpoonup}{\mathrm{r}}=\operatorname{mgd} \cos (0)=(0.55)(9.8)\left(2 \times 10^{-2}\right)=1.1 \times 10^{-1} \mathrm{~J}$

## 5-4- Kinetic energy and the work-kinetic energy theorem

Consider a system consisting of a single object. The figure below shows a block of mass moving through a displacement directed to the right under the action of a net force $\sum \stackrel{\rightharpoonup}{\mathrm{F}}$, also directed to the right. We know from Newton's second law that the block moves with an acceleration $\vec{a}$. If the block moves through a displacement $\Delta \overrightarrow{\mathrm{r}}=\Delta \mathrm{x} \hat{\mathrm{i}}=\left(\mathrm{x}_{\mathrm{f}}-\mathrm{x}_{\mathrm{i}}\right) \hat{\mathrm{i}}$, the net work done on the block by the net force $\sum \overrightarrow{\mathrm{F}}$ is:

$$
\mathrm{W}=\int_{x_{i}}^{x_{f}}\left(\sum F_{x}\right) d x
$$

Using Newton's second law, we can substitute for the magnitude of the net force $\sum \overrightarrow{\mathrm{F}}=\mathrm{m} \overrightarrow{\mathrm{a}}$, and then perform the following chain-rule manipulations on the integrand:

$W=\int_{x_{i}}^{x_{f}} m a d x=\int_{x_{i}}^{x_{f}} m \frac{d v}{d t} d x=\int_{x_{i}}^{x_{f}} m \frac{d v}{d x} \frac{d x}{d t} d x=\int_{v_{i}}^{v_{f}} m v d v=\frac{1}{2} m v_{f}^{2}-\frac{1}{2} m v_{i}^{2}$
$\mathrm{W}=\frac{1}{2} \mathrm{mv}_{\mathrm{f}}^{2}-\frac{1}{2} \mathrm{mv}_{\mathrm{i}}^{2}$
where $v_{i}$ is the speed of the block at $x_{i}$ and $v_{f}$ is its speed at $x_{f}$.
$\mathrm{Eq}(6)$ tells us that the work done by the net force on a particle of mass $m$ is equal to the difference between the initial and final values of a quantity $\frac{1}{2} \mathrm{mv}^{2}$. This quantity is called the kinetic energy K $\mathrm{K}=\frac{1}{2} \mathrm{mv}^{2}$

Kinetic energy is a scalar quantity and has the same units as work. It is often convenient to write eq(6) in the form
$\mathrm{W}=\Delta \mathrm{K}$
Ex: A 6 kg block initially at rest is pulled to the right along a horizontal, frictionless surface by a constant horizontal force of 12 N . Find the speed of the block after it has moved 3 m .

Soln:

$$
\begin{gathered}
W=F \Delta x=(12 \mathrm{~N})(3.0 \mathrm{~m})=36 \mathrm{~J} \\
W=K_{f}-K_{i}=\frac{1}{2} m v_{f}^{2}-0 \\
v_{f}=\sqrt{\frac{2 W}{m}}=\sqrt{\frac{2(36 \mathrm{~J})}{6.0 \mathrm{~kg}}}=3.5 \mathrm{~m} / \mathrm{s}
\end{gathered}
$$



Ex: A block of mass 1.6 kg is attached to a horizontal spring that has a force constant of $1 \times 10^{3} \mathrm{~N} / \mathrm{m}$. The spring is compressed 2 cm and is then released from rest. Calculate the speed of the block as it passes through the equilibrium position $\mathrm{x}=0$ if the surface is frictionless.

Soln:
$\mathrm{W}_{\mathrm{s}}=\frac{1}{2} \mathrm{kx}_{\text {max }}^{2}=\frac{1}{2}\left(10^{3}\right)\left(-2 \times 10^{-2}\right)^{2}=0.2 \mathrm{~J}$
$\mathrm{W}_{\mathrm{s}}=\frac{1}{2} \mathrm{mv}_{\mathrm{f}}^{2}-\frac{1}{2} \mathrm{mv}_{\mathrm{i}}^{2} \quad \rightarrow \quad 0.2=\frac{1}{2} \mathrm{mv}_{\mathrm{f}}^{2}-0=\frac{1}{2}(1.6) \mathrm{v}_{\mathrm{f}}^{2} \quad \rightarrow \quad \mathrm{v}_{\mathrm{f}}=0.5 \mathrm{~m} / \mathrm{s}$

Ex: A particle is subject to a force $F_{x}$ that varies with position as in figure. Find the work done by the force on the particle as it moves (a) from $x=0$ to $x=5 \mathrm{~m}$, (b) from $x=5 \mathrm{~m}$ to $x=10 \mathrm{~m}$, and (c) from $x=10 \mathrm{~m}$ to $x=15 \mathrm{~m}$. (d) What is the total work done by the force over the distance $x=0$ to $x=15 \mathrm{~m}$ ?

Soln:

(a) For the region $0 \leq x \leq 5.00 \mathrm{~m}$,

$$
W=\frac{(3.00 \mathrm{~N})(5.00 \mathrm{~m})}{2}=7.50 \mathrm{~J}
$$

(b) For the region $5.00 \leq x \leq 10.0$,

$$
W=(3.00 \mathrm{~N})(5.00 \mathrm{~m})=15.0 \mathrm{~J}
$$

(c) For the region $10.0 \leq x \leq 15.0$,

$$
W=\frac{(3.00 \mathrm{~N})(5.00 \mathrm{~m})}{2}=7.50 \mathrm{~J}
$$

(d) For the region $0 \leq x \leq 15.0$

$$
W=(7.50+7.50+15.0) \mathrm{J}=30.0 \mathrm{~J}
$$

Ex: A 3 kg object has a velocity $(6 \hat{\mathrm{i}}-2 \hat{\mathrm{j}}) \mathrm{m} / \mathrm{s}$. (a) What is its kinetic energy at this time? (b) Find the total work done on the object if its velocity changes to $(8 \hat{\mathrm{i}}+4 \hat{\mathrm{j}}) \mathrm{m} / \mathrm{s}$.

Soln:
$\mathrm{v}_{\mathrm{i}}^{2}=\overrightarrow{\mathrm{v}}_{\mathrm{i}} \cdot \overrightarrow{\mathrm{v}}_{\mathrm{i}}=36+4=40 \mathrm{~m}^{2} / \mathrm{s}^{2} \quad \rightarrow \quad \mathrm{v}_{\mathrm{i}}=\sqrt{40} \mathrm{~m} / \mathrm{s}$
$\mathrm{K}_{\mathrm{i}}=\frac{1}{2} \mathrm{mv}_{\mathrm{i}}^{2}=\frac{1}{2}(3)(40)=60 \mathrm{~J}$
$\mathrm{v}_{\mathrm{f}}^{2}=\overrightarrow{\mathrm{v}}_{\mathrm{f}} \cdot \overrightarrow{\mathrm{v}}_{\mathrm{f}}=64+16=80 \mathrm{~m}^{2} / \mathrm{s}^{2} \rightarrow \mathrm{v}_{\mathrm{f}}=\sqrt{80} \mathrm{~m} / \mathrm{s}$
$\mathrm{K}_{\mathrm{f}}=\frac{1}{2} \mathrm{mv}_{\mathrm{f}}^{2}=\frac{1}{2}(3)(80)=120 \mathrm{~J}$
$\mathrm{W}=\Delta \mathrm{K} \rightarrow \mathrm{W}=120-60=60 \mathrm{~J}$

