3-4- Uniform Circular Motion

An object moving in a circular path with constant speed, such motion is called uniform circular motion.

In uniform circular motion, we have:

1) The constant-magnitude velocity vector is always tangent to the path of the object and perpendicular to the radius of the circular path.

2) The acceleration vector in uniform circular motion is always perpendicular to the path and always points toward the center of the circle. An acceleration of this nature is called a **centripetal acceleration**, and its magnitude is

$$a_c = \frac{v^2}{r} \quad \text{......................(3-4-1)}$$

where r is the radius of the circle.

To derive equation (3-4-1), consider the diagram of the position and velocity vectors in figure (3-4-1).

The magnitude of the average acceleration is:

$$|\vec{a}_{ave}| = \frac{|\Delta\vec{v}|}{\Delta t} \quad \text{......................(3-4-2)}$$

The two triangles in figure (3-4-1) are similar. This enables us to write a relationship between the lengths of the sides for the two triangles:

$$\frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{r}$$

$$|\Delta\vec{v}| = \frac{v|\Delta\vec{r}|}{r} \quad \text{......................(3-4-3)}$$

Where $v_i = v_f = v$ and $r_i = r_f = r$

Substitute eq. (3-4-3) into eq. (3-4-2)

$$|\vec{a}_{ave}| = \frac{v}{r}\frac{|\Delta\vec{r}|}{\Delta t}$$

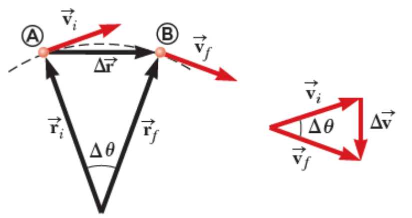


Figure 3-4-1

As Ⓐ and Ⓑ approach each other, Δt approaches zero, and the ratio $|\Delta\vec{r}|/\Delta t$ approaches the speed v. In addition, the average acceleration becomes the instantaneous acceleration at point Ⓐ. Hence, in the limit $\Delta t \to 0$, the magnitude of the acceleration is:

$$a_c = \lim_{\Delta t \to 0} |\vec{a}_{ave}| = \frac{v}{r} \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t} = \frac{v^2}{r}$$

Thus, in uniform circular motion the acceleration is directed inward toward the center of the circle and has magnitude v^2/r.

In many situations it is convenient to describe the motion of a particle moving with constant speed v in a circle of radius r in terms of the period T, which is defined as the time required for one complete revolution. In the time interval T the particle moves a distance of $2\pi r$, which is equal to the circumference of the particle's circular path. Therefore, it follows that:

$$T = \frac{2\pi\, r}{v} \qquad \text{………………….(3-4-4)}$$

3-5- Non-Uniform Circular Motion

If a particle moves along a curved path in such a way that both the magnitude and the direction of velocity vector $\vec{v}$ change in time; we call the motion non-uniform circular motion as shown in figure (3-4-2). So, the particle has an acceleration vector $\vec{a}$ that can be described by two components vectors:

1- The tangential acceleration a_t causes the change in the speed of the particle. It is parallel to the instantaneous velocity and perpendicular to the radius. The magnitude of the tangential acceleration is:

$$a_t = \frac{d|\vec{v}|}{dt} \qquad \text{………………….(3-4-5)}$$

The direction of the tangential component is:

i) In the same direction of $\vec{v}$ (if $\vec{v}$ is increasing).

ii) In the opposite direction to $\vec{v}$ (if $\vec{v}$ is decreasing).

2- The radial (centripetal) acceleration a_r arises from the changes in direction of the velocity vector $\vec{v}$ and has an absolute magnitude given by:

$$a_r = a_c = \frac{v^2}{r}$$

The total acceleration vector $\vec{a}$ can be written as the vector sum of the component vectors:

$$\vec{a} = \vec{a}_t + \vec{a}_r = \frac{d|\vec{v}|}{dt}\hat{\theta} - \frac{v^2}{r}\hat{r} \qquad \text{…………………(3-4-6)}$$

The negative sign indicates that the direction of the centripetal acceleration is toward the center of the circle representing the radius of curvature, which is opposite the direction of the radial unit vector $\hat{r}$, which always points away from the center of the circle.

Because $\vec{a}_r$ and $\vec{a}_t$ are perpendicular component vectors of $\vec{a}$, it follows that the magnitude of $\vec{a}$ is:

$$a = \sqrt{a_r^2 + a_t^2} \qquad \text{…………………(3-4-7)}$$

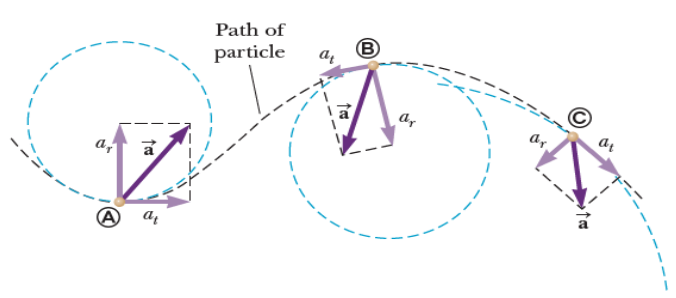
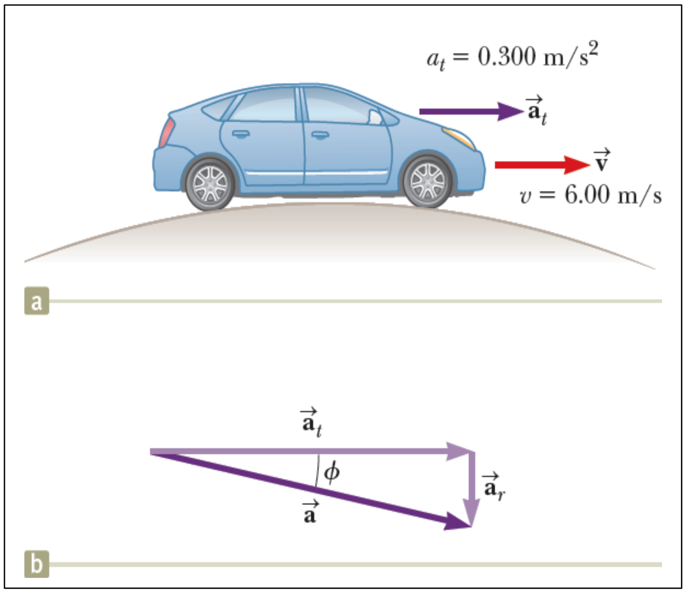


Figure 3-4-2

Ex: A car exhibits a constant acceleration of 0.3 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6 m/s. What is the direction of the total acceleration vector for the car at this instant?

Soln:



$$a_r = -\frac{v^2}{r} = -\frac{(6.00 \text{ m/s})^2}{500 \text{ m}} = -0.072\,0 \text{ m/s}^2$$

$$\sqrt{a_r^2 + a_t^2} = \sqrt{(-0.072\,0 \text{ m/s}^2)^2 + (0.300 \text{ m/s}^2)^2}$$

$$= \boxed{0.309 \text{ m/s}^2}$$

$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.072\,0 \text{ m/s}^2}{0.300 \text{ m/s}^2}\right) = \boxed{-13.5^\circ}$$

Ex: A train slows down as it rounds a sharp horizontal turn, going from 90.0 km/h to 50.0 km/h in 15.0 s that it takes to round the bend. The radius of the curve is 150 m. Compute the acceleration at the moment the train speed reaches 50.0 km/h. Assume the train continues to slow down at this time at the same rate.

Soln:

$$v_i = 90.0 \text{ km/h} = (90.0 \text{ km/h})\ (10^3 \text{ m/km})\ (1 \text{ h}/3\,600 \text{ s}) = 25.0 \text{ m/s}$$

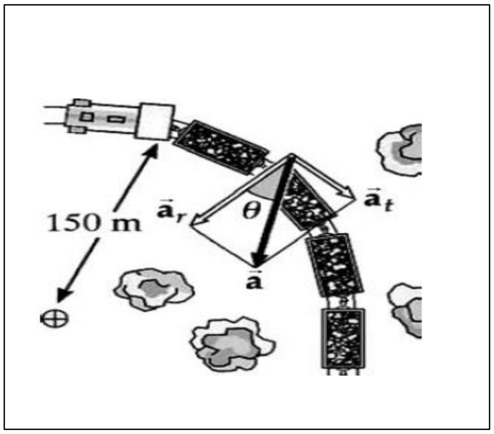
$$v_f = 50.0 \text{ km/h} = (50.0 \text{ km/h})(10^3 \text{ m/km})\ (1 \text{ h}/3\,600 \text{ s}) = 13.9 \text{ m/s}$$

$$a_t = \frac{\Delta v}{\Delta t} = \frac{13.9 \text{ m/s} - 25.0 \text{ m/s}}{15.0 \text{ s}} = -0.741 \text{ m/s}^2 \quad \text{(backward)}$$

and $$a_r = \frac{v^2}{r} = \frac{(13.9 \text{ m/s})^2}{150 \text{ m}} = 1.29 \text{ m/s} \quad \text{(inward)}$$

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(1.29 \text{ m/s}^2)^2 + (-0.741 \text{ m/s}^2)^2} = 1.48 \text{ m/s}^2$$

$$\theta = \tan^{-1}\left(\frac{|a_t|}{a_r}\right) = \tan^{-1}\left(\frac{0.741 \text{ m/s}^2}{1.29 \text{ m/s}^2}\right) = 29.9^\circ$$

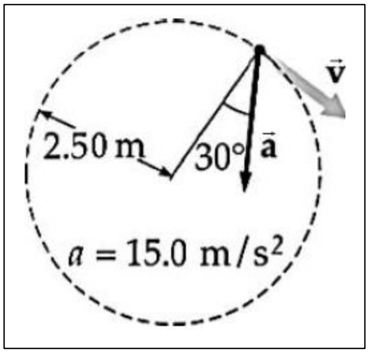


Ex: Figure below represents the total acceleration of a particle moving clockwise in a circle of radius 2.50 m at a certain instant of time. For that instant, find (a) the radial acceleration of the particle, (b) the speed of the particle, and (c) its tangential acceleration. (d) time of period.

Soln:

a) The accelration has a inward radial component:

$a_c = a\cos 30^o = (15)\cos 30^o = 13 \text{ m/s}^2$



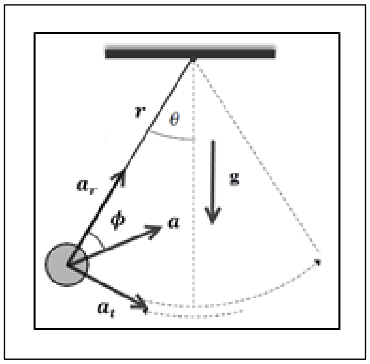
b) $a_c = \dfrac{v^2}{r}$ → $v = \sqrt{a_c r} = \sqrt{(13)(2.5)} = 5.7 \text{ m/s}$

c) The tangential component of the acceleration:

$a_t = a\sin 30^o = (15)\sin 30^o = 7.5 \text{ m/s}^2$

d) $T = \dfrac{2\pi r}{v} = \dfrac{2\pi(2.5)}{5.7} = 2.75 \text{ s}$

Ex: A ball tied to the end of a string of length 0.5 m to swings in a vertical circle under the influence of gravity, when the string makes an angle of $(\theta = 20^\circ)$ with the vertical, the speed of the ball has 1.5 m/s. Find the magnitude of the radial component of acceleration at this instant, the magnitude of the tangential acceleration at the same angle, the magnitude and direction of the total acceleration



Soln:

$a_r = \frac{v^2}{r} = \frac{(1.5)^2 \quad m^2/s^2}{0.5 \quad m} = 4.5\, m/s^2$

$a_t = g\sin(\theta) = 9.8 \times \sin(20^\circ) = 9.8 \times 0.3420 = 3.3518\, m/s^2$

$|\vec{a}| = \sqrt{{a_t}^2 + {a_r}^2} = \sqrt{(3.3518)^2 + (4.5)^2} = 5.6111\, m/s^2$

$\phi = \tan^{-1}\left(\frac{a_t}{a_r}\right) = \tan^{-1}\left(\frac{3.3518}{4.5}\right) = \tan^{-1}(0.7448)$

$\phi = 36.6787^\circ \approx 37^\circ$

Ex: A particle is moving in a horizontal circle path with a radius of 35 cm. It makes 30 revolutions in 12 sec. Find the centripetal acceleration of a particle.
Soln :

The time that particle makes one revolution is: $T = \frac{total\ time}{number\ of\ revolution} = \frac{12}{30} = 0.4\ sec$

The speed of the particle in a horizontal circle path is: $v = \frac{2\pi r}{T} = \frac{2\times\pi\times0.35}{0.4} = 5.4978\ m/s$

The centripetal acceleration of a particle is: $a_c = \frac{v^2}{r} = \frac{(5.4978)^2 \quad m^2/s^2}{0.35 \quad m} = 86.3590\ m/s^2$

Ex: A disk with a radius of 0.3 m is spinning about its central axis at a uniform rate. The velocity of a point on the edge of the disk is 1 m/s. Calculate the times of one period, three period, and centripetal acceleration of a point on this disk located at 9 cm from the axis of rotation, rotating in the time of 0.6 sec.

Soln:
$v = \frac{2\pi r}{T} \Rightarrow T = \frac{2\pi r}{v} = \frac{2\pi\times0.3}{1} = 1.885\ sec$ time of one period.

The time of three period is: $1.885 \times 3 = 5.6549\ sec$

$$v_1 = \frac{2\pi r_1}{T_1} = \frac{2\pi \times 9 \times 10^{-2}}{0.6} = 0.9425\ m/s$$

$$a_r = \frac{{v_1}^2}{r_1} = \frac{(0.9425)^2 \quad m^2/s^2}{9 \times 10^{-2} \quad m} = 9.8701\ m/s^2$$

Ex: A car is driving on a circle path road with radius of 25 m, at speed of 18 m/s. Calculate the period of the car, frequency of moving car and centripetal acceleration.

Soln:

$$v = \frac{2\pi r}{T} \Rightarrow T = \frac{2\pi r}{v} = \frac{2\pi\times25}{18} = 8.7266\ sec$$

$$f = \frac{1}{T} = \frac{1}{8.7266} = 0.1146\ Hz = 1146 \times 10^{-4}\ Hz$$

$$a_r = \frac{v^2}{r} = \frac{(18)^2 \quad m^2/s^2}{25 \quad m} = 12.96\ m/s^2$$