3-3- Projectile Motion

The Projectile moves in a curved path, and its motion is simple to analyze if we make two assumptions: (1) the free-fall acceleration g is constant over the range of motion and is directed downward, and (2) the effect of air resistance is negligible. With these assumptions, we find that the path of a projectile, which we call its trajectory, is always a parabola.

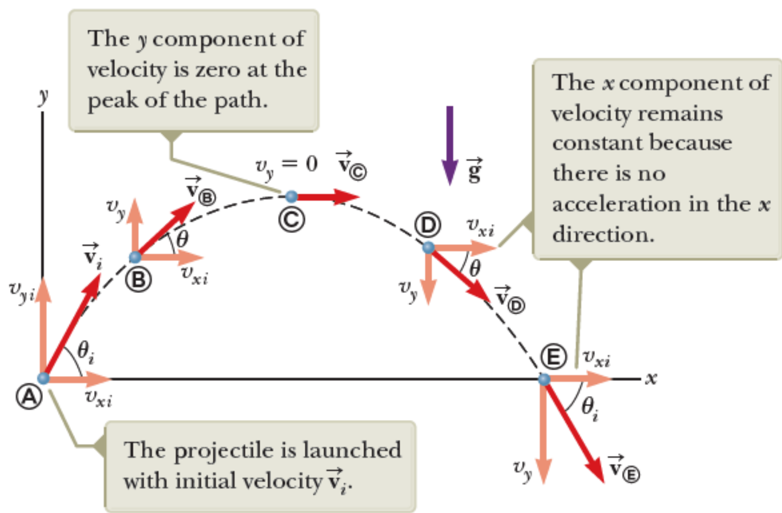


Figure 3.3

When solving projectile motion problems, use three analysis models:

1) The expression for the position vector of the projectile as a function of time is

$$\vec{r}_f = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{g}t^2$$

where the initial *x* and *y* components of the velocity of the projectile are

$v_{xi} = v_i \cos\theta_i$ and $v_{yi} = v_i \sin\theta_i$

2) The particle under constant velocity in the horizontal direction

$$x_f - x_i = v_{xi}t + \frac{1}{2}a_x t^2 \quad \rightarrow \quad x_f = x_i + v_i \cos\theta_i t$$

3) The particle under constant acceleration in the vertical direction with a_y =-g :

$$v_{yf} = v_{yi} - gt \quad \rightarrow \quad v_{yf} = v_i \sin\theta_i - gt$$

$$y_f - y_i = v_{yi}t - \frac{1}{2}gt^2 \quad \rightarrow \quad y_f - y_i = v_i \sin\theta_i t - \frac{1}{2}gt^2$$

$$v_{yf}^2 = v_{yi}^2 - 2g(y_f - y_i) \quad \rightarrow \quad v_{yf}^2 = v_i^2 \sin^2\theta_i - 2g(y_f - y_i)$$

Ex: Show that the trajectory of a projectile is a parabola.

Soln.

Assume a projectile is launched from the origin $x_i = y_i = 0$

$$x_f = x_i + v_i \cos\theta_i t \quad \rightarrow \quad x_f = v_i \cos\theta_i t \qquad \text{..........(1)}$$

$$y_f - y_i = v_{yi}t - \frac{1}{2}gt^2 \quad \rightarrow \quad y_f = v_i \sin\theta_i t - \frac{1}{2}gt^2 \qquad \text{..........(2)}$$

Insert equation (1) into equation (2)

$$y = v_i \sin\theta_i \frac{x}{v_i \cos\theta_i} - \frac{1}{2}g\left(\frac{x}{v_i \cos\theta_i}\right)^2$$

$$y = (\tan\theta_i)x - \left(\frac{g}{2v_i^2 \cos^2\theta_i}\right)x^2$$

The equation is of the form $y = ax - bx^2$, which is the equation of a parabola that passes through the origin.

3-3-1- Horizontal range and maximum height of a projectile:

Assume a projectile is launched from the origin at t_i=0, with a positive v_{yi} component as shown in figure 3.4 and returns to the same horizontal level.

Two points in this motion are especially interesting to analyze: the peak point Ⓐ, which has Cartesian coordinates (R/2, h), and the point Ⓑ, which has coordinates (R, 0). The distance R is called the horizontal range of the projectile, and the distance h is its maximum height. Let us find h and R mathematically in terms of v_i, θ_i, and g.

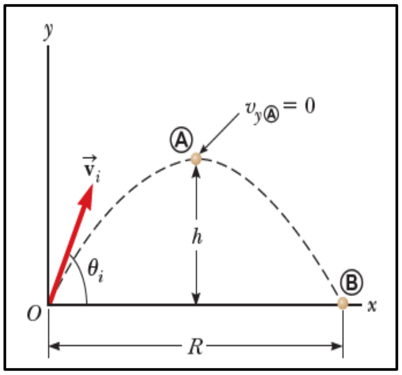


Figure 3.4

$$v_{yf} = v_{yi} - gt \quad \rightarrow \quad 0 = v_i \sin\theta_i - gt_{Ⓐ}$$

$$t_{Ⓐ} = \frac{v_i \sin\theta_i}{g}$$

$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2 \quad \rightarrow \quad h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \tfrac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{v_i^2 \sin^2\theta_i}{2g}$$

The range R is the horizontal position of the projectile at a time that is twice the time at which it reaches its peak, that is, at time $t_B = 2t_A$.

$$x_f = x_i + v_{xi}t \quad \rightarrow \quad R = v_{xi}t_{Ⓑ} = (v_i \cos\theta_i)2t_{Ⓐ}$$

$$= (v_i \cos\theta_i)\frac{2v_i \sin\theta_i}{g} = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g}$$

Using $\sin 2\theta = 2\sin\theta\cos\theta$

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

The maximum value of R from the above equation is $R_{max} = v_i^2/g$. The maximum value of $\sin 2\theta$ is 1, which occurs when $2\theta = 90^o$. Therefore, R is a maximum when $\theta = 45^o$.

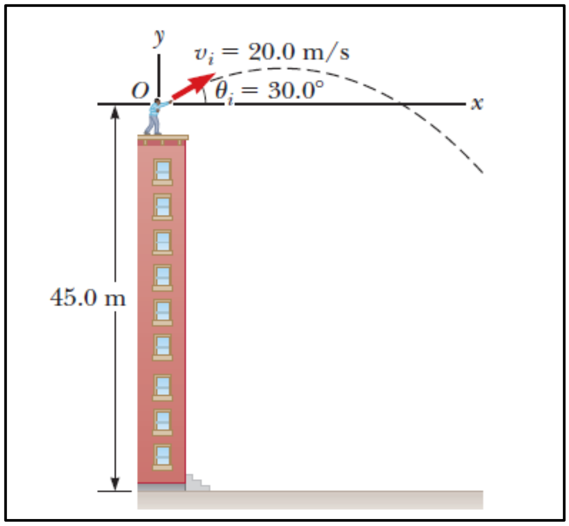
Ex: A long jumper leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s. (A) How far does he jump in the horizontal direction? (B) What is the maximum height reached?

Soln:

$$R = \frac{v_i^2 \sin 2\theta_i}{g} = \frac{(11.0\ \mathrm{m/s})^2 \sin 2(20.0^\circ)}{9.80\ \mathrm{m/s^2}} = \boxed{7.94\ \mathrm{m}}$$

$$h = \frac{v_i^2 \sin^2\theta_i}{2g} = \frac{(11.0\ \mathrm{m/s})^2(\sin 20.0^\circ)^2}{2(9.80\ \mathrm{m/s^2})} = \boxed{0.722\ \mathrm{m}}$$

Ex: A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s. The height from which the stone is thrown is 45.0 m above the ground. (A) How long does it take the stone to reach the ground? (B) What is the speed of the stone just before it strikes the ground?



Soln.

(A) $v_{xi} = v_i \cos\theta_i = (20 \text{ m/s})\cos30^o = 17.3 \text{ m/s}$

$v_{yi} = v_i \sin\theta_i = (20 \text{ m/s})\sin30^o = 10.0 \text{ m/s}$

$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$

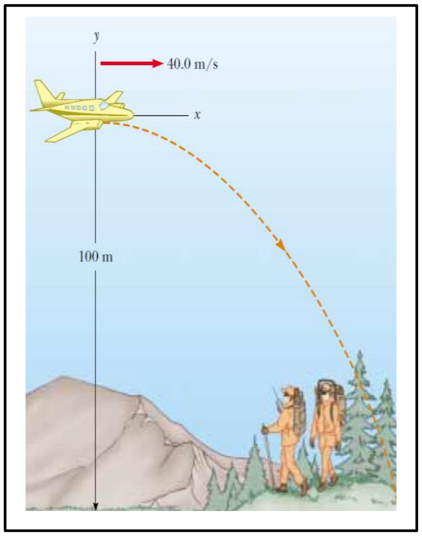
$-45 \text{ m} = 0 + (10 \text{ m/s})t - \frac{1}{2}\left(9.8 \text{ m/s}^2\right)t^2 \rightarrow$ t=4.22 s

(B) $v_{yf} = v_{yi} - gt$

$v_{yf} = (10 \text{ m/s})\text{-}\left(9.8 \text{ m/s}^2\right)(4.22 \text{ s}) = -31.3 \text{ m/s}$

$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(17.3 \text{ m/s})^2 + (31.3 \text{ m/s})^2} = 35.8 \text{ m/s}$

Ex: A plane drops a package of supplies to a party of explorers, as shown in figure below. If the plane is traveling horizontally at 40.0 m/s and is 100 m above the ground. A) where does the package strike the ground relative to the point at which it is released? B) what are the horizontal and vertical component of the velocity of the package just before hits the ground. C) Where is the plane when the package hits the ground (assume the plane doesn't change its speed)



Soln.

A) $x_f = x_i + v_{xi}t \rightarrow x_f = 0 + (40)t \rightarrow x_f = 40t$

$y_f - y_i = v_{yi}t - \frac{1}{2}gt^2 \rightarrow -100 - 0 = 0 - \frac{1}{2}(9.8)t^2$

$t = 4.52 \text{ s} \rightarrow x_f = (40)(4.52) = 181 \text{ m}$

B) The horizontal component of the velocity of the package remain constant → $v_{xf} = v_{xi} = 40 \text{ m/s}$

$v_{yf}^2 = v_{yi}^2 - 2g\Delta y \rightarrow v_{yf}^2 = 0 - 2\left(9.8 \text{ m/s}^2\right)(-100 \text{ m}) \rightarrow v_{yf} = \pm 44.3 \text{ m/s}$

Take $v_{yf} = -44.3 \text{ m/s}$ because directed downward

C) $x_f = v_x t \rightarrow x_f = (40)(4.52) = 181 \text{ m}$ (directly over the package)

Ex: A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection?

Soln.

We want to find θ_i such that $R = 3h$. We can use the equations for the range and height of a projectile's trajectory,

$$R = \left(v_i^2 \sin 2\theta_i\right)/g \qquad \text{and} \qquad h = \left(v_i^2 \sin^2 \theta_i\right)/2g$$

We combine different requirements by mathematical substitution into $R = 3h$, thus:

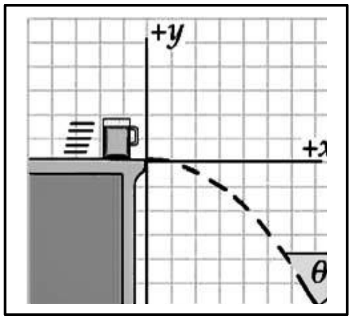
$$\frac{v_i^2 \sin 2\theta_i}{g} = \frac{3v_i^2 \sin^2 \theta_i}{2g} \qquad \text{or} \qquad \frac{2}{3} = \frac{\sin^2 \theta_i}{\sin 2\theta_i}$$

But $\sin 2\theta_i = 2 \sin \theta_i \cos \theta_i$ so $\dfrac{\sin^2 \theta_i}{\sin 2\theta_i} = \dfrac{\sin^2 \theta_i}{2 \sin \theta_i \cos \theta_i} = \dfrac{\tan \theta_i}{2}$

Substituting and solving for θ_i gives $\theta_i = \tan^{-1}\left(\dfrac{4}{3}\right) = 53.1°$

Ex: In a local restaurant, a customer slides an empty mug down the counter for a refill. The height of the counter is 1.22 m. The mug slides off the counter and strikes the floor 1.40 m from the base of the counter. (a) With what velocity did the mug leave the counter?
(b) What was the direction of the mug's velocity just before it hit the floor?

Soln.



Vertical motion: $y_f = -1.22$ m $\quad v_{yi} = 0 \quad v_y = ? \quad a_y = -9.80$ m/s^2

Horizontal motion: $x_f = 1.40$ m $\quad v_x = ? =$ constant $\quad a_x = 0$

(a) To find the time interval of fall, we use the equation for motion with constant acceleration $y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$

Substituting, $-1.22 \text{ m} = 0 + 0 - \frac{1}{2}(9.80 \text{ m/s}^2)\, t^2$

so $t = [2(1.22 \text{ m})\text{s}^2/9.80 \text{ m}]^{0.5} = 0.499$ s

Then $v_x = \dfrac{x_f}{t} = \dfrac{1.40 \text{ m}}{0.499 \text{ s}} = 2.81$ m/s

(b) The mug hits the floor with a vertical velocity of $v_{yf} = v_{yi} + a_y t$ and an impact angle below the horizontal of $\theta = \tan^{-1}(v_{yf}/v_x)$

Evaluating v_{yf}, $v_{yf} = 0 - (9.80 \text{ m/s}^2)(0.499 \text{ s}) = -4.89$ m/s

Thus, $\theta = \tan^{-1}\left(\dfrac{-4.89 \text{ m/s}}{2.81 \text{ m/s}}\right) = -60.2° = 60.2°$ below the horizontal.