## Chapter 3: Motion in two dimensions

## 3-1- The Position, Velocity, and Acceleration Vectors

In chapter 2 we found that the motion of a particle moving along a straight line is completely known if its position is known as a function of time. Now let us extend this idea to motion in the xy-plane. We begin by describing the position of a particle by its position vector $\overrightarrow{\mathrm{r}}$, drawn from the origin of some coordinate system to the particle located in the xy-plane, as in figure 3.1.

The displacement vector $\Delta \overrightarrow{\mathrm{r}}$ for the particle is defined as the difference between its final position vector and its initial position vector

$$
\begin{equation*}
\Delta \overrightarrow{\mathrm{r}}=\overrightarrow{\mathrm{r}}_{\mathrm{f}}-\overrightarrow{\mathrm{r}}_{\mathrm{i}} \tag{3-1-1}
\end{equation*}
$$

The average velocity $\overrightarrow{\mathrm{v}}_{\text {ave }}$ of a particle during the time interval $\Delta t$ is defined as the displacement of the particle divided by the time interval.
$\overrightarrow{\mathrm{v}}_{\text {ave }}=\frac{\Delta \stackrel{\rightharpoonup}{\mathrm{r}}}{\Delta \mathrm{t}}$
Note that the average velocity between points is independent of the path taken. This is because average velocity is proportional to displacement, which depends only on the initial and final position vectors and not on the path taken.


Figure 3.1


Figure 3.2

The instantaneous velocity $\overrightarrow{\mathrm{v}}$ is defined as the limit of the average velocity $\Delta \overrightarrow{\mathrm{r}} / \Delta \mathrm{t}$ as $\Delta \mathrm{t}$ approaches zero
$\overrightarrow{\mathrm{v}}=\lim _{\Delta \mathrm{t} \rightarrow 0} \frac{\Delta \overrightarrow{\mathrm{r}}}{\Delta \mathrm{t}}=\frac{\mathrm{d} \overrightarrow{\mathrm{r}}}{\mathrm{dt}}$

That is, the instantaneous velocity equals the derivative of the position vector with respect to time.

The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion as shown in figure 3.2.

The magnitude of the instantaneous velocity vector $v=|\vec{v}|$ is called the speed, which is a scalar quantity.

The average acceleration of a particle as it moves is defined as the change in the instantaneous velocity vector $\Delta \overrightarrow{\mathrm{v}}$ divided by the time interval $\Delta \mathrm{t}$ during which that change occurs
$\overrightarrow{\mathrm{a}}_{\mathrm{ave}}=\frac{\Delta \overrightarrow{\mathrm{v}}}{\Delta \mathrm{t}}=\frac{\stackrel{\rightharpoonup}{\mathrm{v}}_{\mathrm{f}}-\overrightarrow{\mathrm{v}}_{\mathrm{i}}}{\mathrm{t}_{\mathrm{f}}-\mathrm{t}_{\mathrm{i}}}$
The instantaneous acceleration $a$ is defined as the limiting value of the ratio $\Delta \vec{v} / \Delta t$ as $\Delta t$ approaches zero

$$
\begin{equation*}
\stackrel{\rightharpoonup}{\mathrm{a}}=\lim _{\Delta \mathrm{t} \rightarrow 0} \frac{\Delta \overrightarrow{\mathrm{v}}}{\Delta \mathrm{t}}=\frac{\mathrm{d} \overrightarrow{\mathrm{v}}}{\mathrm{dt}} \tag{3-1-5}
\end{equation*}
$$

## 3-2- Two-Dimensional Motion with Constant Acceleration

The position vector for a particle moving in the xy-plane can be written

$$
\overrightarrow{\mathrm{r}}=x \hat{\mathrm{i}}+\mathrm{y} \hat{\mathrm{j}}
$$

If the position vector is known, the velocity of the particle can be obtained

$$
\begin{equation*}
\vec{v}=\frac{d \vec{r}}{d t}=v_{x} \hat{i}+v_{y} \hat{j} \tag{3-2-2}
\end{equation*}
$$

Substituting, $v_{x f}=v_{x i}+a_{x} t$ and $v_{y f}=v_{y i}+a_{y} t$ into the equation (3-2-2), we will obtain
$\vec{v}_{f}=\left(v_{x i}+a_{x} t\right) \hat{i}+\left(v_{y i}+a_{y} t\right) \hat{j} \quad \vec{v}_{f}=\left(v_{x i} \hat{i}+v_{y i} \hat{j}\right)+\left(a_{x} t \hat{i}+a_{y} \hat{t}\right)$
$\vec{v}_{f}=\vec{v}_{i}+\vec{a} t$

Similarly, substituting both $x_{f}-x_{i}=v_{x i} t+\frac{1}{2} a_{x} t^{2}$ and $y_{f}-y_{i}=v_{y i} t+\frac{1}{2} a_{y} t^{2}$ into equation (3-2-1)

$$
\begin{align*}
& \left(x_{\mathrm{f}} \hat{\mathrm{i}}+y_{\mathrm{f}} \hat{\mathrm{j}}\right)=\left(\mathrm{x}_{\mathrm{i}} \hat{\mathrm{i}}+y_{\mathrm{i}} \hat{\mathrm{j}}\right)+\left(\mathrm{v}_{\mathrm{xi}} \hat{\mathrm{i}}+v_{\mathrm{yi}} \hat{\mathrm{j}}\right) \mathrm{t}+\frac{1}{2}\left(\mathrm{a}_{\mathrm{x}} \hat{\mathrm{i}}+\mathrm{a}_{\mathrm{y}} \hat{\mathrm{j}}\right) \mathrm{t}^{2} \\
& \overrightarrow{\mathrm{r}}_{\mathrm{f}}=\overrightarrow{\mathrm{r}}_{\mathrm{i}}+\vec{v}_{\mathrm{i}} \mathrm{t}+\frac{1}{2} \overrightarrow{\mathrm{a}} \mathrm{t}^{2} \tag{3-2-4}
\end{align*}
$$

Ex: A particle moves in the xy-plane, starting from the origin at $t=0$ with an initial velocity having an $x$-component of $20 \mathrm{~m} / \mathrm{s}$ and a y-component of $-15 \mathrm{~m} / \mathrm{s}$. The particle experiences an acceleration in the x-direction, given by $4.0 \mathrm{~m} / \mathrm{s}^{2}$. (A) Determine the total velocity vector at any time. (B) Calculate the velocity and speed of the particle at $t=5.0 \mathrm{~s}$ and the angle the velocity vector makes with the $x$ axis. (C) Determine the $x$ and $y$ coordinates of the particle at any time $t$ and its position vector at this time.

Soln:
We have: $\quad v_{x i}=20 \mathrm{~m} / \mathrm{s}, \quad v_{y i}=-15 \mathrm{~m} / \mathrm{s} \quad, \quad a_{x}=4.0 \mathrm{~m} / \mathrm{s}^{2}$ and $a_{y}=0 \mathrm{~m} / \mathrm{s}^{2}$
(A)
$\vec{v}_{f}=\vec{v}_{i}+\vec{a} t=\left(v_{x i} \hat{i}+v_{y i} \hat{j}\right)+\left(a_{x} t \hat{i}+a_{y} \hat{t} \hat{j}\right)$
$\vec{v}_{\mathrm{f}}=(20 \hat{\mathrm{i}}-15 \hat{\mathrm{j}})+(4 t \hat{\mathrm{i}}+(0) \hat{\mathrm{t}}) \quad \rightarrow \quad \overrightarrow{\mathrm{v}}_{\mathrm{f}}=[(20+4 \mathrm{t}) \hat{\mathrm{i}}-15 \hat{\mathrm{j}}] \mathrm{m} / \mathrm{s}$
(B)
$\overrightarrow{\mathrm{v}}_{\mathrm{f}}=[(20+4(5)) \hat{\mathrm{i}}-15 \hat{\mathrm{j}}] \mathrm{m} / \mathrm{s}=[40 \hat{\mathrm{i}}-15 \hat{\mathrm{j}}] \mathrm{m} / \mathrm{s}$
speed $\equiv\left|\overrightarrow{\mathrm{v}}_{\mathrm{f}}\right|=\mathrm{v}_{\mathrm{f}}=\sqrt{\mathrm{v}_{\mathrm{xf}}^{2}+\mathrm{v}_{\mathrm{yf}}^{2}}=\sqrt{(40)^{2}+(15)^{2}}=43 \mathrm{~m} / \mathrm{s}$
$\theta=\tan ^{-1}\left(\frac{\mathrm{v}_{\mathrm{yf}}}{\mathrm{v}_{\mathrm{xi}}}\right)=\tan ^{-1}\left(\frac{-15}{40}\right)=-21^{\circ}$
(C)
$x_{f}-x_{i}=v_{x i} t+\frac{1}{2} a_{x} t^{2} \quad \rightarrow \quad x_{f}=\left(20 t+2 t^{2}\right) m$

$$
\begin{array}{lll}
y_{f}-y_{i}=v_{y i} t+\frac{1}{2} a_{y} t^{2} & \rightarrow & y_{f}=(-15 t) m \\
\vec{r}=x \hat{i}+y \hat{j} & \rightarrow & \vec{r}=\left[\left(20 t+2 t^{2}\right) \hat{i}-15 \hat{t}\right] \mathrm{m}
\end{array}
$$

Ex: A motorist drives south at $20 \mathrm{~m} / \mathrm{s}$ for 3 min , then turns west and travels at $25 \mathrm{~m} / \mathrm{s}$ for 2 min , and finally travels northwest at $30 \mathrm{~m} / \mathrm{s}$ for 1 min . For this 6 min trip, find (a) the total vector displacement, (b) the average speed, and (c) the average velocity.

Soln.
a)

$$
\begin{aligned}
& \Delta \overrightarrow{\mathrm{r}}=\overrightarrow{\mathrm{v}}_{1} \mathrm{t}_{1}+\overrightarrow{\mathrm{v}}_{2} \mathrm{t}_{2}+\overrightarrow{\mathrm{v}}_{3} \mathrm{t}_{3} \\
& \Delta \overrightarrow{\mathrm{r}}=(20 \mathrm{~m} / \mathrm{s})(180 \mathrm{~s})(-\hat{\mathrm{j}})+(25 \mathrm{~m} / \mathrm{s})(120 \mathrm{~s})(-\hat{\mathrm{i}})+\left[(30 \mathrm{~m} / \mathrm{s})(60 \mathrm{~s}) \cos \left(45^{\circ}\right)(-\hat{\mathrm{i}})+(30 \mathrm{~m} / \mathrm{s})(60 \mathrm{~s}) \sin \left(45^{\circ}\right)(\hat{\mathrm{j}})\right] \\
& \Delta \overrightarrow{\mathrm{r}}=-(3600 \mathrm{~m}) \hat{\mathrm{j}}-(3000 \mathrm{~m}) \hat{\mathrm{i}}+[-(1273 \mathrm{~m}) \hat{\mathrm{i}}+(1273 \mathrm{~m}) \hat{\mathrm{j}}] \\
& \Delta \overrightarrow{\mathrm{r}}=[-(4273) \hat{\mathrm{i}}-(2327) \hat{\mathrm{j}}] \mathrm{m}=[-(4.27) \hat{\mathrm{i}}-(2.33) \hat{\mathrm{j}}] \mathrm{km}
\end{aligned}
$$

Also, the answer above can be written as:
$|\Delta \overrightarrow{\mathrm{r}}|=\Delta \mathrm{r}=\sqrt{(-4.27)^{2}+(-2.33)^{2}}=4.87 \mathrm{~km}$ at $\theta=\tan ^{-1}\left(\frac{2.33}{4.27}\right)=28.6^{\circ}$ south of west $\underline{\mathrm{or}} \theta=209^{\circ}$ from the east.
b)

The total distance or the path-length traveled is:
$\mathrm{d}=(20 \mathrm{~m} / \mathrm{s})(180 \mathrm{~s})+(25 \mathrm{~m} / \mathrm{s})(120 \mathrm{~s})+(30 \mathrm{~m} / \mathrm{s})(60 \mathrm{~s})=8400 \mathrm{~m}=8.4 \mathrm{~km}$
average speed $=\frac{\mathrm{d}}{\mathrm{t}}=\frac{8.4 \mathrm{~km}}{6 \mathrm{~min}}=23.3 \mathrm{~m} / \mathrm{s}$
c)
$\overrightarrow{\mathrm{v}}_{\text {ave }}=\frac{\Delta \overrightarrow{\mathrm{r}}}{\mathrm{t}}=\frac{[-(4.27) \hat{\mathrm{i}}-(2.33) \hat{\mathrm{j}}] \mathrm{km}}{6 \mathrm{~min}}=[-(11.9) \hat{\mathrm{i}}-(6.47) \hat{\mathrm{j}}] \mathrm{m} / \mathrm{s}$

Ex: A fish swimming in a horizontal plane has velocity $\vec{v}_{i}=(4 \hat{i}+\hat{j}) \mathrm{m} / \mathrm{s}$ at a point in the ocean where the position relative to a certain rock is $\overrightarrow{\mathrm{r}}_{\mathrm{i}}=(10 \hat{\mathrm{i}}-4 \hat{\mathrm{j}}) \mathrm{m}$. After the fish swims with constant acceleration for 20 s , its velocity is $\overrightarrow{\mathrm{v}}=(20 \hat{\mathrm{i}}-5 \hat{\mathrm{j}}) \mathrm{m} / \mathrm{s}$. (a) What are the components of the acceleration of the fish? (b) What is the direction of the acceleration with respect to unit vector $\hat{\mathrm{i}}$ ? (c) If the fish maintains constant acceleration, where is it at $t=25 \mathrm{~s}$, and in what direction is it moving.

## Soln.

a)
$\mathrm{a}_{\mathrm{x}}=\frac{\Delta \mathrm{v}_{\mathrm{x}}}{\Delta \mathrm{t}}=\frac{[20-4] \mathrm{m} / \mathrm{s}}{20 \mathrm{~s}}=0.8 \mathrm{~m} / \mathrm{s}^{2}$
$\mathrm{a}_{\mathrm{y}}=\frac{\Delta \mathrm{v}_{\mathrm{y}}}{\Delta \mathrm{t}}=\frac{[-5-1] \mathrm{m} / \mathrm{s}}{20 \mathrm{~s}}=-0.3 \mathrm{~m} / \mathrm{s}^{2}$
b)
$\theta=\tan ^{-1}\left(\frac{\mathrm{a}_{\mathrm{y}}}{\mathrm{a}_{\mathrm{x}}}\right)=\tan ^{-1}\left(\frac{-0.3}{0.8}\right)=-20.6^{\circ} \quad$ or $\quad \theta=339^{\circ}$ from the +x -axis
c)
$\mathrm{x}_{\mathrm{f}}-\mathrm{x}_{\mathrm{i}}=\mathrm{v}_{\mathrm{xi}} \mathrm{t}+\frac{1}{2} \mathrm{a}_{\mathrm{x}} \mathrm{t}^{2} \quad \rightarrow \quad \mathrm{x}_{\mathrm{f}}=\left[10+(4)(25)+\frac{1}{2}(0.8)(25)^{2}\right]=360 \mathrm{~m}$
$y_{f}-y_{i}=v_{y i} t+\frac{1}{2} a_{y} t^{2} \quad \rightarrow \quad y_{f}=\left[-4+(1)(25)+\frac{1}{2}(-0.3)(25)^{2}\right]=-72.8 m$
$\overrightarrow{\mathrm{r}}_{\mathrm{f}}=\mathrm{x}_{\mathrm{f}} \hat{\mathrm{i}}+\mathrm{y}_{\mathrm{f}} \hat{\mathrm{j}} \quad \rightarrow \quad \overrightarrow{\mathrm{r}}_{\mathrm{f}}=[360 \hat{\mathrm{i}}-72.8 \hat{\mathrm{j}}] \mathrm{m}$
$\mathrm{v}_{\mathrm{xf}}=\mathrm{v}_{\mathrm{xi}}+\mathrm{a}_{\mathrm{x}} \mathrm{t} \quad \rightarrow \quad \mathrm{v}_{\mathrm{xf}}=4+(0.8)(25)=24 \mathrm{~m} / \mathrm{s}$
$\mathrm{v}_{\mathrm{yf}}=\mathrm{v}_{\mathrm{yi}}+\mathrm{a}_{\mathrm{y}} \mathrm{t} \quad \rightarrow \quad \mathrm{v}_{\mathrm{yf}}=1+(-0.3)(25)=-6.5 \mathrm{~m} / \mathrm{s}$
$\theta=\tan ^{-1}\left(\frac{\mathrm{v}_{\mathrm{yf}}}{\mathrm{v}_{\mathrm{xi}}}\right)=\tan ^{-1}\left(\frac{-6.5}{24}\right)=-15.2^{\circ}$ or $\theta=345^{\circ}$ from the +x -axis.

