Chapter 1: Vectors

1-1- Systems of units

In mechanics, there are three basic quantities: length, mass, and time. All other quantities in mechanics can be expressed in terms of these three.

In 1960, an international committee established a set of standards for the fundamental quantities of science. It is called the SI (Système International), and its fundamentals units of length, mass, and time are the meter, kilogram, and second, respectively. Other SI standards established by the committee are those for temperature (the kelvin), electric current (the ampere), luminous intensity (the candela), and the amount of substance (the mole).

1-2- Coordinate Systems

1-Cartesian coordinates (rectangular coordinates)

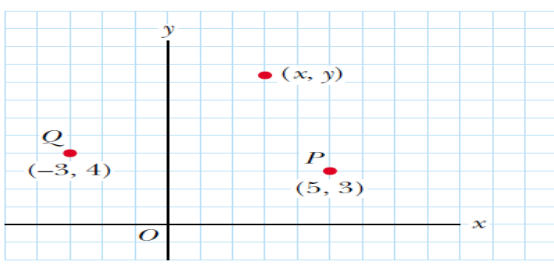


Figure 1.1: Designation of points in a Cartesian coordinate system. Every point is labeled with coordinates (*x*, *y*).

2-plane polar coordinates (r, θ)

Sometimes it is more convenient to represent a point in a plane by its plane polar coordinates (r,θ) as shown in figure 1.2(a). In this polar coordinate system, r is the distance from the origin to the point having Cartesian coordinates (x, y) and θ is the angle between a fixed axis and a line drawn from the origin to the point. The fixed axis is often the positive *x* axis, and θ is usually measured counterclockwise from it.

We can obtain relationship between Cartesian coordinates and polar coordinates from figure 1.2(b):

❖ Cartesian coordinates in terms of polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

❖ Polar coordinates in terms of Cartesian coordinates

$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

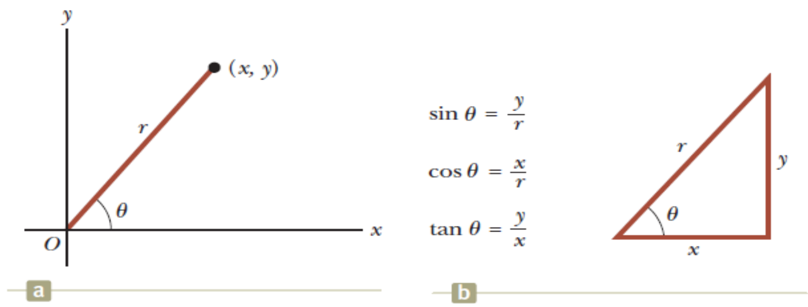


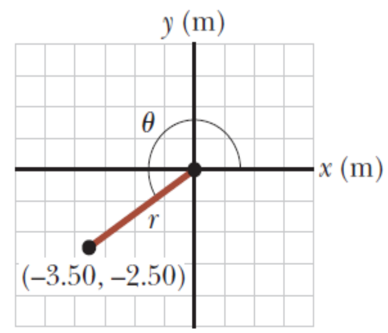
Figure 1.2 (a) The plane polar coordinates of a point are represented by the distance *r* and the angle θ, where θ is measured counterclockwise from the positive *x* axis. (b) The right triangle used to relate (*x*, *y*) to (*r*, θ).

EX: The Cartesian coordinates of a point in the xy-plane are (x, y)=(-3.50, -2.50)m. Find the polar coordinates of this point.

Soln.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.5)^2 + (-2.5)^2} = 4.3\text{m}$$

$$\tan\theta = \frac{y}{x} = \frac{-2.5}{-3.5} = 0.714 \;\rightarrow\; \theta = \tan^{-1}(0.714) = 216^{o}$$



1-3- Scalar and vector quantities

A scalar quantity is specified by a single value with an appropriate unit and has no direction.

Examples of scalar quantities are volume, mass, speed, temperature, and time intervals.

A vector quantity is completely specified by a number and appropriate units plus a direction.

Examples of vector quantities are displacement, velocity, momentum, and force.

1-4- Some properties of vectors

1- Equality of two vectors:

A two vectors $\vec{A}$ and $\vec{B}$ may be defined to be equal if they have the same magnitude and point in the same direction.

2- Adding vectors:

The rules for adding vectors are conveniently described by geometric methods. To add vector $\vec{B}$ to vector $\vec{A}$, first draw vector $\vec{A}$, with its magnitude represented by a convenient scale, on graph paper and then draw vector $\vec{B}$ to the same scale with its tail starting from the tip of $\vec{A}$, as shown in the figure 1.3. The resultant vector $\vec{R}=\vec{A}+\vec{B}$ is the vector drawn from the tail of $\vec{A}$ to the tip of $\vec{B}$. This procedure is known as the triangle method of addition.

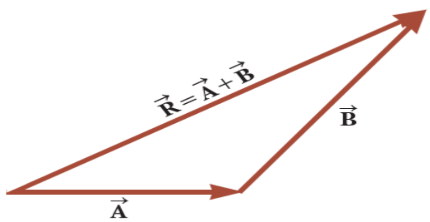


Figure 1.3: When vector $\vec{B}$ is added to vector $\vec{A}$, the resultant $\vec{R}$ is the vector that runs from the tail of $\vec{A}$ to the tip of $\vec{B}$.

- The property $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ is called commutative law of addition.

- The property $\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$ is called associative law of addition as shown in figure 1.4.

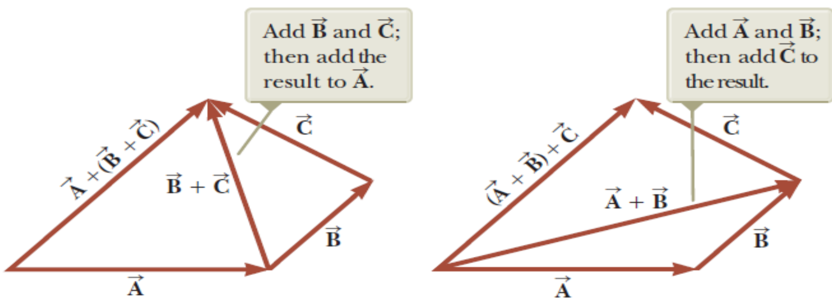


Figure 1.4: Geometric constructions for verifying the associative law of addition.

❖ A geometric construction can also be used to add more than two vectors. This is shown in figure 1.5 for the case of four vectors. The resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$ is the vector that completes the polygon. In other words, $\vec{R}$ is the vector drawn from the tail of the first vector to the tip of the last vector. This technique for adding vectors is often called the "head to tail method."

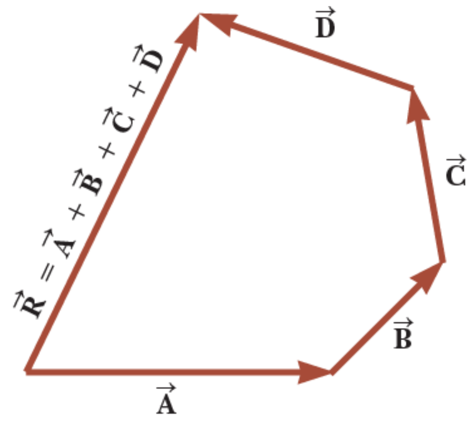


Figure 1.5: Geometric construction for summing four vectors.
The resultant vector $\vec{R}$ is by definition the one that completes the polygon.

❖ An alternative graphical procedure for adding two vectors, known as the parallelogram rule of addition, is shown in figure 1.6. The tails of the two vectors $\vec{A}$ and $\vec{B}$ are joined together and the resultant vector $\vec{R}$ is the diagonal of a parallelogram formed with $\vec{A}$ and $\vec{B}$ as two of its four sides.

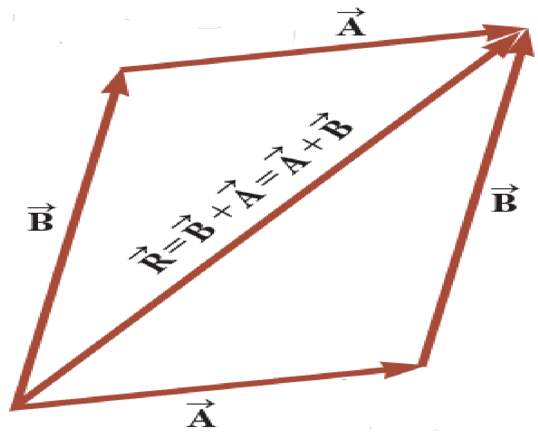


Figure 1.6

3- Negative of a vector:

The negative of the vector $\vec{A}$ is defined as the vector that when added to $\vec{A}$ gives zero for the vector sum. That is, $\vec{A}+(-\vec{A})=0$. The vectors $\vec{A}$ and $-\vec{A}$ have the same magnitude but point in opposite directions.

4- Subtracting vectors:

❖ We define the operation $\vec{A}-\vec{B}$ as vector $-\vec{B}$ added to vector $\vec{A}$,

$$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$$

The geometric construction for subtracting two vectors in this way is illustrated in figure 1.7(a).

❖ Another way of looking at vector subtraction, is to notice that the difference $\vec{A}-\vec{B}$ points from the tip of the second vector to the tip of the first, as shown in figure 1.7(b).

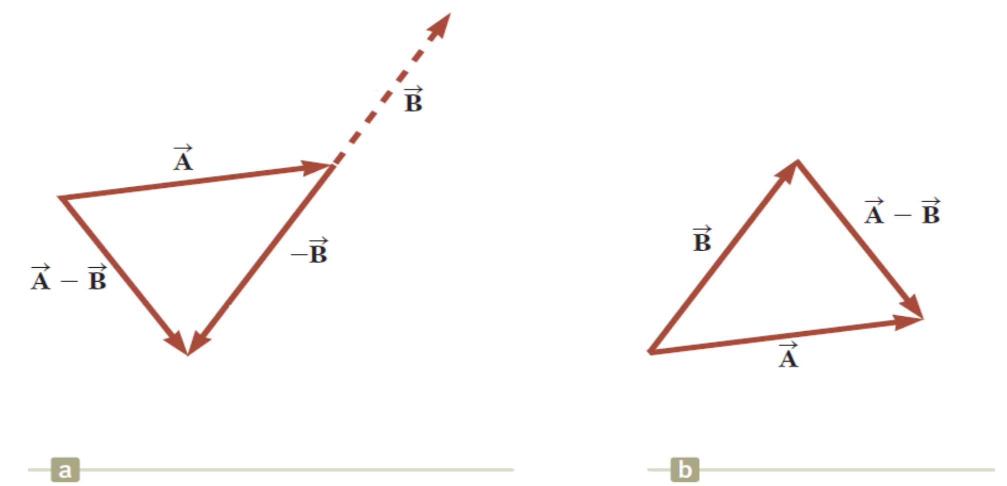


Figure 1.7: (a) Subtracting vector $\vec{B}$ from vector $\vec{A}$. The vector $-\vec{B}$ is equal in magnitude to vector $\vec{B}$ and points in the opposite direction. (b) A second way of looking at vector subtraction.

5- Multiplying a vector by a scalar

If vector $\vec{A}$ is multiplied by a positive scalar quantity m, the product $m\vec{A}$ is a vector that has the same direction as $\vec{A}$ and magnitude mA. If vector $\vec{A}$ is multiplied by a negative scalar quantity -m, the product $-m\vec{A}$ is directed opposite $\vec{A}$.

1-5- Components of a vector and unit vectors

Consider a vector $\vec{A}$ lying in the xy-plane and making an arbitrary angle θ with the positive x-axis as shown in figure 1.8. This vector can be expressed as the sum of two components A_x, which is parallel to the x-axis, and A_y, which is parallel to the y-axis.

The components of $\vec{A}$ are:

$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$

The magnitude and direction of $\vec{A}$ are related to its components through the expressions:

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

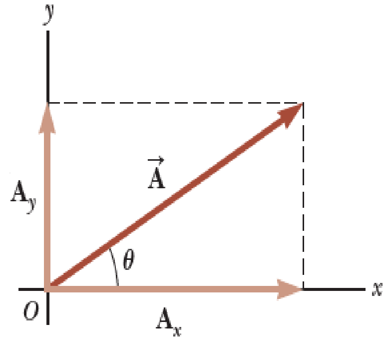


Figure 1.8: A vector $\vec{A}$ lying in the xy-plane can be represented by its component vectors A_x and A_y.

The vector $\vec{A}$ can be written in unit vector notation:

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

Where $\hat{i}$ and $\hat{j}$ are called unit vector, a dimensionless vector having a magnitude of 1.

$$|\hat{i}| = |\hat{j}| = 1$$

Suppose we wish to add vector $\vec{B}$ to vector $\vec{A}$ where vector $\vec{B}$ has components B_x and B_y. The resultant vector $\vec{R} = \vec{A} + \vec{B}$ is therefore

$$\vec{R} = \left(A_x\hat{i} + A_y\hat{j}\right) + \left(B_x\hat{i} + B_y\hat{j}\right)$$

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

the components of the resultant vector are

$$R_x = (A_x + B_x)$$
$$R_y = (A_y + B_y)$$

The magnitude of $\vec{R}$ and the angle θ it makes with the x-axis are obtained from its components using the relationships:

$$R = \sqrt{R_x^2 + R_y^2}$$

$$R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{(A_y + B_y)}{(A_x + B_x)}$$

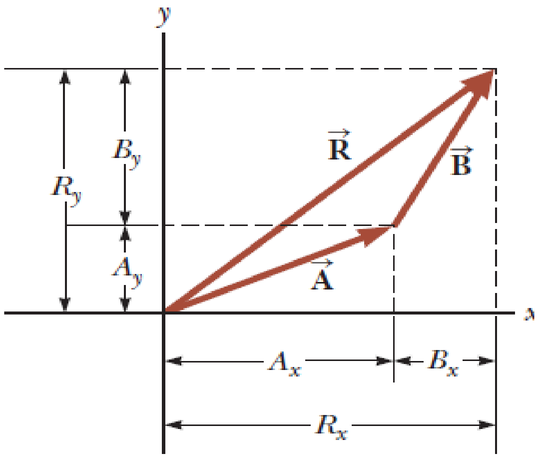


Figure 1.9: Geometric construction for the sum of two vectors shows the relationship between the components of the resultant $\vec{R}$ and the components of the individual vectors.

If $\vec{A}$ and $\vec{B}$ both have x, y, and z components, we express them in the form

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

The sum of $\vec{A}$ and $\vec{B}$

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

The magnitude of the vector $\vec{R}$ is:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}$$

Ex: Find the sum of two displacement vectors $\vec{A}$ and $\vec{B}$ lying in the xy-plane and given by

$\vec{A} = \left(2\hat{i} + 2\hat{j}\right)$ m and $\vec{B} = \left(2\hat{i} - 4\hat{j}\right)$ m.

Soln.

$$\vec{R} = \vec{A} + \vec{B} = \left(2\hat{i} + 2\hat{j}\right) + \left(2\hat{i} - 4\hat{j}\right) = \left(4\hat{i} - 2\hat{j}\right) \text{ m}$$

$$R_x = 4 \text{ m} \quad , \quad R_y = -2 \text{ m}$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4)^2 + (-2)^2} = 4.5 \text{ m}$$

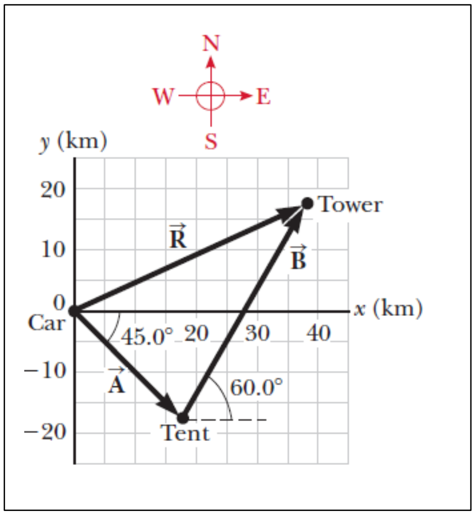
$$\tan\theta = \frac{R_y}{R_x} = \frac{(-2)}{(4)} = -0.5 \quad \rightarrow \quad \theta = \tan^{-1}(-0.5) = 333^o$$

Ex: A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

a) Determine the components of the hiker's displacement for each day.

b) Determine the components of the hiker's resultant displacement $\vec{R}$ for the trip. Find an expression for $\vec{R}$ in terms of unit vectors.

Soln:



a)

$A_x = A\cos\left(-45^o\right)$ = (25 km)(0.707) = 17.7 km

$A_y = A\sin\left(-45^o\right)$ = (25 km)(- 0.707) = - 17.7 km

$B_x = B\cos\left(60^o\right)$ = (40 km) (0.5) = 20 km

$B_y = B\sin\left(60^o\right)$ = (40 km) (0.866) = 34.6 km

b)

$R_x = A_x + B_x$ = 17.7 km + 20 km = 37.7 km

$R_y = A_y + B_y$ = - 17.7 km + 34.6 km = 17 km

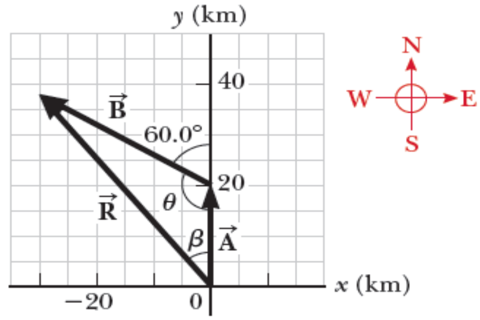
$\vec{R} = \left(37.7\hat{i} + 17\hat{j}\right)$ km

Ex: A car travels 20.0 km due north and then 35.0 km in a direction 60° west of north. Find the magnitude and direction of the car's resultant displacement.

Soln :

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$$R = \sqrt{(20.0\text{ km})^2 + (35.0\text{ km})^2 - 2(20.0\text{ km})(35.0\text{ km})\cos 120°}$$
$$= 48.2\text{ km}$$



$$\frac{\sin\beta}{B} = \frac{\sin\theta}{R}$$

$$\sin\beta = \frac{B}{R}\sin\theta = \frac{35.0\text{ km}}{48.2\text{ km}}\sin 120° = 0.629$$

$$\beta = 38.9°$$

The resultant displacement of the car is 48.2 km in a direction 38.9° west of north.

1-6- The Scalar product of two vectors

The scalar product of any two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity equal to the product of the magnitudes of the two vectors and the cosine of the angle θ between them:

$$\vec{A}.\vec{B} = AB\cos\theta$$

Scalar product is commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

Scalar product obeys the distributive law of multiplication:

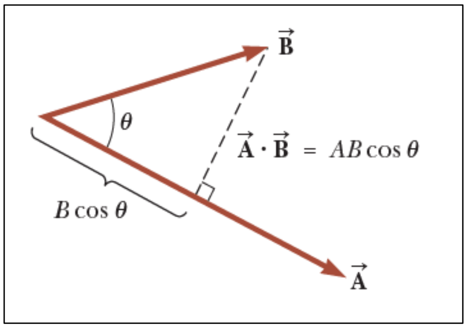
$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

Because of the dot symbol, the scalar product is often called the dot product.

The scalar products of the unit vectors are:

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$



Consider two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

The scalar product of $\vec{A}$ and $\vec{B}$:

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

If $\vec{A} = \vec{B}$

$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2$

Ex: The vectors $\vec{A}$ and $\vec{B}$ are given by $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = -\hat{i} + 2\hat{j}$

a) Determine the scalar product $\vec{A}$ and $\vec{B}$.
b) Find the angle between $\vec{A}$ and $\vec{B}$.

Soln

$$\vec{A}\cdot\vec{B} = \left(2\hat{i} + 3\hat{j}\right)\cdot\left(-\hat{i} + 2\hat{j}\right)$$

$$\vec{A}\cdot\vec{B} = -2\hat{i}\cdot\hat{i} + 2\hat{i}\cdot 2\hat{j} - 3\hat{j}\cdot\hat{i} + 3\hat{j}\cdot 2\hat{j}$$

$$= -2(1) + 4(0) - 3(0) + 6(1) = 4$$

$$A = \sqrt{A_x^2 + A_y^2} \quad \rightarrow \quad A = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$B = \sqrt{B_x^2 + B_y^2} \quad \rightarrow \quad B = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} \quad \rightarrow \quad \cos\theta = \frac{4}{\sqrt{13}\sqrt{5}} = \frac{4}{\sqrt{65}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{65}}\right) = 60.3^{o}$$

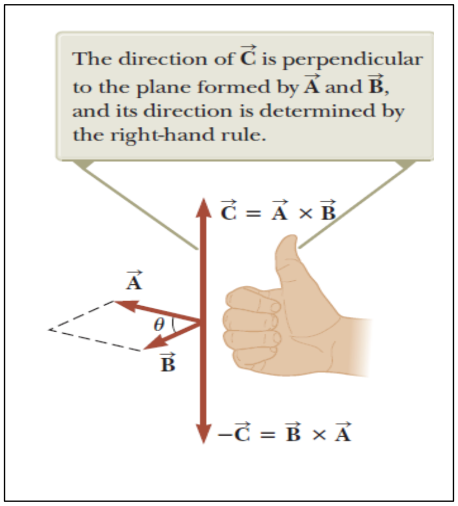
1-7-The Vector product

The vector product $\vec{A}\times\vec{B}$ between any two vectors $\vec{A}$ and $\vec{B}$, is defined as a third vector $\vec{C}$, which has a magnitude of $AB\sin\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

$$\vec{A}\times\vec{B}=\vec{C}$$

Its magnitude is: $C=AB\sin\theta$

Because of the notation, $\vec{A}\times\vec{B}$ is often read "$\vec{A}$ cross $\vec{B}$" so the vector product is also called the cross product.



Vector product has the following properties:

1. The vector product is not commutative

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

2- If $\vec{A}$ is parallel to $\vec{B}$, then $\theta=0$ or 180^{o}; therefore, it follows that:

$$\vec{A}\times\vec{B}=0$$

3- If $\vec{A}$ is perpendicular to $\vec{B}$, then $\theta=90^{o}$; therefore, it follows that:

$$\left|\vec{A}\times\vec{B}\right|=AB$$

4- The vector product obeys distributive law:

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

5- The derivative of the vector product with respect to some variable such as t is:

$$\frac{d}{dt}\left(\vec{A}\times\vec{B}\right)=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

Cross products of unit vectors obey the following rules :

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$$
$$\hat{i}\times\hat{j}=-\hat{j}\times\hat{i}=\hat{k}$$
$$\hat{j}\times\hat{k}=-\hat{k}\times\hat{j}=\hat{i}$$
$$\hat{k}\times\hat{i}=-\hat{i}\times\hat{k}=\hat{j}$$

The cross product of any two vectors $\vec{A}$ and $\vec{B}$, can be expressed in the following determinant form:

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}\hat{i}+\begin{vmatrix}A_z & A_x\\ B_z & B_x\end{vmatrix}\hat{j}+\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\hat{k}$$

Expanding these determinants gives the result

$$\vec{A}\times\vec{B}=\left(A_yB_z-A_zB_y\right)\hat{i}+\left(A_zB_x-A_xB_z\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

Ex: The vectors $\vec{A}$ and $\vec{B}$ are given by $\vec{A}=2\hat{i}+3\hat{j}$ and $\vec{B}=-\hat{i}+2\hat{j}$. Find $\vec{A}\times\vec{B}$ and verify that $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$.

Soln.

$$\vec{A}\times\vec{B}=\left(2\hat{i}+3\hat{j}\right)\times\left(-\hat{i}+2\hat{j}\right)$$

$$\vec{A}\times\vec{B}=\left(2\hat{i}\right)\times\left(-\hat{i}\right)+\left(2\hat{i}\right)\times\left(2\hat{j}\right)+\left(3\hat{j}\right)\times\left(-\hat{i}\right)+\left(3\hat{j}\right)\times\left(2\hat{j}\right)$$

$$\vec{A}\times\vec{B}=0+4\hat{k}+3\hat{k}+0=7\hat{k}$$

$$\vec{B}\times\vec{A}=\left(-\hat{i}+2\hat{j}\right)\times\left(2\hat{i}+3\hat{j}\right)$$

$$\vec{B}\times\vec{A}=\left(-\hat{i}\right)\times\left(2\hat{i}\right)+\left(-\hat{i}\right)\times\left(3\hat{j}\right)+\left(2\hat{j}\right)\times\left(2\hat{i}\right)+\left(2\hat{j}\right)\times\left(3\hat{j}\right)$$

$$\begin{aligned}\vec{B}\times\vec{A}&=0-3\hat{k}-4\hat{k}+0=-7\hat{k}\\ &=-\vec{A}\times\vec{B}\end{aligned}$$