3) Since $\lim_{j\to\infty} a_j = \lim_{j\to\infty} \frac{j}{5j+3}$

$= \lim_{j\to\infty} \frac{1}{5+\frac{3}{j}} = \frac{1}{5+0} = \frac{1}{5} \neq 0$. Then the

series $\sum_{j=1}^{\infty} \frac{j}{5j+3}$ diverges by theorem 15.14

(the nth-term test for divergence).

Exercises 15.19: Determine whether each of the following series converges or diverges:

1) $\sum_{i=0}^{\infty} 2\left(\tan\frac{\pi}{4}\right)^i$,

2) $\sum_{i=1}^{\infty} \frac{7(-1)^i}{5^i}$.

Theorem 15.20: If $\sum_{n=1}^{\infty} a_n$ converges, then

$\lim_{n\to\infty} a_n = 0$.

Warning: Theorem 15.20 does not say that, if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. The series $\sum_{n=1}^{\infty} a_n$ may diverge even though $\lim_{n\to\infty} a_n = 0$.

Example 15.21: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ (by the following theorem).

Theorem 15.22: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Definition 15.23: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series.

Example 15.24: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series which is convergent by theorem 15.22

Theorem 15.25: If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ exist and $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ (where A and B are finite), then

i) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$,

ii) $\sum_{n=1}^{\infty} k.a_n = k \sum_{n=1}^{\infty} a_n = k.A$ (where k is any number).

Corollary 15.26: If $\sum_{n=1}^{\infty} a_n$ diverges and c is any nonzero number then $\sum_{n=1}^{\infty} c a_n$ diverges.

Definition 15.27: The series $\sum_{n=1}^{\infty} (a_n + b_n)$ is called the sum of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.
The series $\sum_{n=1}^{\infty} (a_n - b_n)$ is called the difference of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

Example 15.28: Calculate the sum of each of the series $\sum_{n=1}^{\infty} \frac{4}{2^{n-1}}$ and $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{7^n}$ if the series converge.

Solution:

1) $\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = 4 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 4 \cdot \frac{1}{1-\frac{1}{2}}$
$= 4 \cdot \frac{1}{\frac{1}{2}} = 4 . 2 = 8$.

2) $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{3^n}{7^n} - \frac{2^n}{7^n} \right)$

$= \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n - \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$

$= \sum_{m=1}^{\infty} \left(\frac{3}{7}\right)^{m-1} - \sum_{m=1}^{\infty} \left(\frac{2}{7}\right)^{m-1}$ (where $m = n+1$)

$= \frac{1}{1-\frac{3}{7}} - \frac{1}{1-\frac{2}{7}} = \frac{1}{\frac{4}{7}} - \frac{1}{\frac{5}{7}}$

$= \frac{7}{4} - \frac{7}{5} = \frac{35-28}{20} = \frac{7}{20}$.

Remark 15.29: If $\sum_{n=1}^{\infty} a_n$ converges and k is an index greater than 1. Then $\sum_{n=k}^{\infty} a_n$ converges and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

which implies that

$$\sum_{n=k}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - (a_1 + a_2 + \cdots + a_{k-1})$$

$$= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{k-1} a_n .$$

Example 15.30: Calculate the sum of each of the series $\sum_{i=3}^{\infty} \frac{1}{5^i}$ and $\sum_{i=4}^{\infty} \frac{1}{2^i}$ if the series converge.

Solution:

1) $\sum_{i=3}^{\infty} \frac{1}{5^i} = \sum_{i=1}^{\infty} \frac{1}{5^i} - \left(\frac{1}{5} + \frac{1}{5^2}\right)$

$= \sum_{i=1}^{\infty} \left(\frac{1}{5} \cdot \left(\frac{1}{5}\right)^{i-1}\right) - \left(\frac{1}{5} + \frac{1}{5^2}\right)$

$= \frac{\frac{1}{5}}{1-\frac{1}{5}} - \frac{6}{25} = \frac{\frac{1}{5}}{\frac{4}{5}} - \frac{6}{25}$

$= \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{4} - \frac{6}{25} = \frac{1}{4} - \frac{6}{25}$

$= \frac{25-24}{100} = \frac{1}{100}.$

2) $\sum_{i=4}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i} - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right)$

$= \sum_{i=1}^{\infty} \left(\frac{1}{2} \cdot \left(\frac{1}{2}\right)^{i-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)$

$= \frac{\frac{1}{2}}{1-\frac{1}{2}} - \frac{7}{8} = \frac{\frac{1}{2}}{\frac{1}{2}} - \frac{7}{8} = 1 - \frac{7}{8} = \frac{1}{8}.$

Remark 15.31: The index of terms of a series can be changed without changing the convergence of the series.

Examples 15.32:

1) $$\sum_{n=0}^{\infty} \frac{3^n-2^n}{7^n} = \sum_{n=1}^{\infty} \frac{3^{n-1}-2^{n-1}}{7^{n-1}}$$

$$= \sum_{n=5}^{\infty} \frac{3^{n-5}-2^{n-5}}{7^{n-5}} = \frac{7}{20}$$ (see Ex. 15.28 (2) for the sum of the series).

2) $$\sum_{i=3}^{\infty} \frac{1}{5^i} = \sum_{i=5}^{\infty} \frac{1}{5^{i-2}} = \sum_{i=1}^{\infty} \frac{1}{5^{i+2}}$$

$$= \frac{1}{100}$$ (see Example 15.30 (1) for the sum of the series).

§16: Tests for Convergence of Series

1) Comparison Test for Series of Nonnegative Terms

Let $\sum_{n=1}^{\infty} a_n$ be a series that has no negative terms, Then

1) the series $\sum_{n=1}^{\infty} a_n$ converges if there is a convergent series of nonnegative terms $\sum_{n=1}^{\infty} c_n$ with $a_n \leq c_n$ for all $n \geq n_0$ for some integer n_0,

2) the series $\sum_{n=1}^{\infty} a_n$ diverges if there is a divergent series of nonnegative terms $\sum_{n=1}^{\infty} d_n$ with $a_n \geq d_n$ for all $n \geq n_0$ for some integer n_0.

Example 16.1: The series $\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots$

$$= \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$$

converges since all its terms are positive and for all $n \geq 4$, the nth terms are less than the corresponding terms of the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$$

$$+ \cdots = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

Remark 16.2: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e \cong 2.71828.$

Example 16.3: Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ converges.

Solution: Since $0 < \dfrac{n}{n+1} < 1$ for all positive integers n, then $\dfrac{n}{2^n(n+1)} < \dfrac{1}{2^n}$ for all positive integers n.

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$

$$= \frac{1}{2^1 \cdot 2} + \frac{2}{2^2 \cdot 3} + \frac{3}{2^3 \cdot 4} + \frac{4}{2^4 \cdot 5} + \cdots$$

$= \frac{1}{4} + \frac{1}{6} + \frac{3}{32} + \frac{1}{20} + \cdots$ converges since all its terms are positive and less than the corresponding terms of the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}}$$

$= 1$.

Example 16.4: Determine whether the series $\sum_{i=1}^{\infty} \frac{i}{5i^2-4}$ converges or diverges.

Solution :

Since $\frac{i}{5i^2-4} > \frac{i}{5i^2} = \frac{1}{5i} = \frac{1}{5}\left(\frac{1}{i}\right)$.

Then the series $\sum_{i=1}^{\infty} \frac{i}{5i^2-4} = \frac{1}{1} + \frac{2}{16} + \frac{3}{41} + \frac{4}{76} + \cdots = 1 + \frac{1}{8} + \frac{3}{41} + \frac{1}{19} + \cdots$

diverges since all its terms are positive and greater than the corresponding terms of the series

$\sum_{i=1}^{\infty} \frac{1}{5}\left(\frac{1}{i}\right) = \frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \cdots$ which diverges by corollary 15.26 since $\sum_{i=1} \frac{1}{i}$ diverges by theorem 15.22.