with $a = \frac{1}{9}$ and $r = \frac{1}{3}$, which is by theorem 15.7 converges and

$$\sum_{i=1}^{\infty} \frac{1}{3^{i+1}} = \frac{\frac{1}{9}}{(1-\frac{1}{3})} = \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{\cancel{9}_3} \cdot \frac{\cancel{3}}{2} = \frac{1}{6}.$$

Example 15.9: Find $\sum_{i=1}^{\infty} (-\frac{1}{2})^{i-3}$ if the series $\sum_{i=1}^{\infty} (-\frac{1}{2})^{i-3}$ converges.

Solution:
The series $\sum_{i=1}^{\infty} (-\frac{1}{2})^{i-3}$

$$= \sum_{i=1}^{\infty} \left(-\frac{1}{2}\right)^{-2} \left(-\frac{1}{2}\right)^{i-1}$$

$$= \sum_{i=1}^{\infty} \frac{1}{(-2)^{-2}} \left(-\frac{1}{2}\right)^{i-1}$$

$$= \sum_{i=1}^{\infty} (-2)^2 \left(-\frac{1}{2}\right)^{i-1} = \sum_{i=1}^{\infty} 4 \left(-\frac{1}{2}\right)^{i-1}$$ is

a geometric series with $a = 4$ and $r = -\frac{1}{2}$, which is by theorem 15.7 converges and $\sum_{i=1}^{\infty} (-\frac{1}{2})^{i-3} = \frac{4}{1-(-\frac{1}{2})}$

$$= \frac{4}{\frac{3}{2}} = 4 \cdot \frac{2}{3} = \frac{8}{3}.$$

Example 15.10: Find the geometric series with $a=5$, $r=\frac{1}{3}$, and its sum.

Solution: The series is $5+5\left(\frac{1}{3}\right)+5\left(\frac{1}{3}\right)^2$

$$+5\left(\frac{1}{3}\right)^3+\cdots=5+\frac{5}{3}+\frac{5}{9}+\frac{5}{27}+\cdots$$

$=\sum_{i=1}^{\infty}5\left(\frac{1}{3}\right)^{i-1}$ and the sum is

$$\frac{a}{1-r}=\frac{5}{1-\frac{1}{3}}=\frac{5}{\frac{2}{3}}=5\cdot\frac{3}{2}=\frac{15}{2}.$$

Example 15.11: Find $\sum_{i=1}^{\infty}\frac{1}{5^{i+2}}$.

Solution:

$\sum_{i=1}^{\infty}\frac{1}{5^{i+2}}=\sum_{i=1}^{\infty}\frac{1}{5^3}\cdot\frac{1}{5^{i-1}}$ which is a geometric series with $a=\frac{1}{5^3}$ and $r=\frac{1}{5}$. Thus $\sum_{i=1}^{\infty}\frac{1}{5^{i+2}}=\frac{a}{1-r}$

$$=\frac{\frac{1}{5^3}}{1-\frac{1}{5}}=\frac{\frac{1}{5^3}}{\frac{4}{5}}=\frac{1}{5^3}\cdot\frac{5}{4}=\frac{1}{5^2\cdot 4}=\frac{1}{100}.$$

Example 15.12: Find $\sum_{i=0}^{\infty}\frac{2}{6^i}$.

Solution: $\sum_{i=0}^{\infty}\frac{2}{6^i}=2+\frac{2}{6}+\frac{2}{6^2}+\cdots$

$=\sum_{i=1}^{\infty}\frac{2}{6^{i-1}}$ which is a geometric series with $a = 2$ and $r = \frac{1}{6}$.

Thus $\sum_{i=0}^{\infty} \frac{2}{6^i} = \frac{a}{1-r} = \frac{2}{1-\frac{1}{6}} = \frac{2}{\frac{5}{6}}$

$= 2 \cdot \frac{6}{5} = \frac{12}{5} = 2.4$.

Example 15.13: Find $\sum_{i=3}^{\infty} \frac{7}{2^i}$.

Solution: $\sum_{i=3}^{\infty} \frac{7}{2^i} = \frac{7}{8} + \frac{7}{16} + \frac{7}{32} + \cdots$

$= \frac{7}{8} + \frac{7}{8}\left(\frac{1}{2}\right) + \frac{7}{8}\left(\frac{1}{2}\right)^2 + \cdots$

$= \sum_{i=1}^{\infty} \frac{7}{8}\left(\frac{1}{2}\right)^{i-1}$ which is a geometric series with $a = \frac{7}{8}$ and $r = \frac{1}{2}$.

Thus $\sum_{i=3}^{\infty} \frac{7}{2^i} = \frac{a}{1-r} = \frac{\frac{7}{8}}{1-\frac{1}{2}} = \frac{\frac{7}{8}}{\frac{1}{2}}$

$= \frac{7}{8} \cdot \frac{2}{1} = \frac{7}{4} = 1.75$.

Theorem 15.14 (The nth-term Test for Divergence):

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 15.15: Determine whether the series $\sum_{n=1}^{\infty} n^2$ converges or diverges.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty \neq 0.$$

Thus by theorem 15.14, the series $\sum_{n=1}^{\infty} n^2$ diverges.

Example 15.16: Determine whether the series $\sum_{i=1}^{\infty} \frac{i+3}{i}$ converges or diverges.

Solution:

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \left(\frac{i+3}{i}\right) = \lim_{i\to\infty} \left(1 + \frac{3}{i}\right)$$

$$= \lim_{i\to\infty} 1 + \lim_{i\to\infty} \frac{3}{i} = \lim_{i\to\infty} 1 + 3 \cdot \lim_{i\to\infty} \frac{1}{i}$$

$$= 1 + 3(0) = 1 + 0 = 1 \neq 0.$$

Thus by theore 15.14, the series $\sum_{i=1}^{\infty} \frac{i+3}{i}$ diverges.

Example 15.17 : Determine whether the series $\sum_{j=1}^{\infty} (-1)^{j+1}$ converges or diverges.

Solution :

$\lim_{j \to \infty} a_j = \lim_{j \to \infty} (-1)^{j+1}$ does not exist

(since $\lim_{\substack{j \to \infty \\ j \text{ is odd}}} (-1)^{j+1} = 1$ and $\lim_{\substack{j \to \infty \\ j \text{ is even}}} (-1)^{j+1}$

$= -1$).

Thus by theorem 15.14 , the series

$\sum_{j=1}^{\infty} (-1)^{j+1}$ diverges.

Solved Exercises 15.18 : Determine whether each of the following series converges or diverges :

1) $\sum_{i=1}^{\infty} \frac{1}{i^2+i}$,

2) $\sum_{n=1}^{\infty} 3 \left(\cos \frac{\pi}{3}\right)^n$,

3) $\sum_{j=1}^{\infty} \frac{j}{5j+3}$.

Solution:

1) Since $\frac{1}{i^2+i} = \frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$,

then $\sum_{i=1}^{\infty} \frac{1}{i^2+i} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right)$.

Thus the partial sum $s_n = \sum_{i=1}^{n} \frac{1}{i^2+i}$

$= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right)$

$+ \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.

Therefore $\sum_{i=1}^{\infty} \frac{1}{i^2+i} = \lim_{n\to\infty} s_n$

$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1}$

$= 1 - \lim_{m\to\infty} \frac{1}{m}$ (where $m = n+1$)

$= 1 - 0 = 1$.

Thus the series $\sum_{i=1}^{\infty} \frac{1}{i^2+i}$ converges.

2) The series $\sum_{n=1}^{\infty} 3\left(\cos\frac{\pi}{3}\right)^n$

$= \sum_{n=1}^{\infty} \left(3\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}\right)^{n-1}$ is a geometric

series with $a = 3\cos\frac{\pi}{3} = 3 \cdot \frac{1}{2} = \frac{3}{2}$,

and $r = \cos\frac{\pi}{3} = \frac{1}{2}$. Thus the series

converges and its sum is $\frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$.