Definition 14.45: A sequence $\{a_n\}$ of real numbers is said to be

1. increasing if $a_n \leq a_{n+1}$ $\forall n \geq 1$,
2. decreasing if $a_n \geq a_{n+1}$ $\forall n \geq 1$,
3. Strictly increasing if $a_n < a_{n+1}$ $\forall n \geq 1$,
4. Strictly decreasing if $a_n > a_{n+1}$ $\forall n \geq 1$.

Example 14.46: The sequence $\{\frac{1}{n}\}$ is strictly decreasing since $n < n+1$

$$\Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow a_n > a_{n+1} \quad \forall n \geq 1.$$

Example 14.47: The sequence $\{n+2\}$ is strictly increasing since $n+2 < n+3 = (n+1)+2$
$\Rightarrow a_n < a_{n+1}$ $\forall n \geq 1$

Example 14.48: The sequence $\{a_n\}$ defined by $a_n = \begin{cases} n^2 & \text{if } n = 1,2 \\ n+1 & \text{if } n \geq 3 \end{cases}$ is increasing
since $a_1 = 1, a_2 = 4, a_3 = 4, a_4 = 5, a_5 = 6, \ldots$
$\Rightarrow a_n \leq a_{n+1}$ $\forall n \geq 1$.

Example 14.49: The sequence $\left\{\frac{1}{n^2 - 3n + 3}\right\}$ is decreasing since $a_1 = 1, a_2 = 1, a_3 = \frac{1}{3}$,
$a_4 = \frac{1}{7}, a_5 = \frac{1}{13}, a_6 = \frac{1}{21}, \ldots \Rightarrow a_n \geq a_{n+1}$ $\forall n \geq 1$.

§15: Infinite Series

Definition 15.1: If $\{a_n\}$ is a given sequence, then an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{i=1}^{\infty} a_i$$

is called an infinite series (or series).

1. The number a_n is called the nth term of the series.

2. The number $s_n = \sum_{i=1}^{n} a_i$ is called the nth partial sum of the series $\sum_{i=1}^{\infty} a_i$.

3. The partial sums $s_1, s_2, s_3, \ldots, s_n, \ldots$ of the series $\sum_{i=1}^{\infty} a_i$ form a sequence $\{s_n\}$ called the sequence of the partial sums of the infinite series $\sum_{i=1}^{\infty} a_i$.

4. If the sequence of the partial sums $\{s_n\}$ has a limit s (i.e. $\lim_{n\to\infty} s_n = s$), we say that the series $\sum_{i=1}^{\infty} a_i$ converges to s and we write $\sum_{i=1}^{\infty} a_i = s$, otherwise if the sequence $\{s_n\}$ has no limit, we

say that the series $\sum_{i=1}^{\infty} a_i$ diverges (i.e. the series $\sum_{i=1}^{\infty} a_i$ converges to s if and only if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = s$).

If $\sum_{i=1}^{\infty} a_i = s$, then s is called the sum of the series $\sum_{i=1}^{\infty} a_i$.

Example 15.2: Let $\{a_i\}$ be the sequence defined by

$$a_i = \begin{cases} i & \text{if } i = 1,2 \\ \frac{1}{10^{i-2}} & \text{if } i > 2 \end{cases}$$

which means that $a_1 = 1$, $a_2 = 2$, $a_3 = \frac{1}{10^{3-2}} = \frac{1}{10}$, $a_4 = \frac{1}{10^{4-2}} = \frac{1}{10^2}$, $a_5 = \frac{1}{10^{5-2}} = \frac{1}{10^3}$, ...

Then the partial sum of the infinite series $\sum_{i=1}^{\infty} a_i$ will be as follows:

$$s_1 = \sum_{i=1}^{1} a_i = a_1 = 1 \quad ,$$

$$s_2 = \sum_{i=1}^{2} a_i = a_1 + a_2 = 1 + 2 = 3 \quad ,$$

$$s_3 = \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = 1 + 2 + \frac{1}{10} = 3.1 \quad ,$$

$$S_4 = \sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4 = 1 + 2 + \frac{1}{10} + \frac{1}{100} = 3.11 ,$$

$$S_5 = \sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 2 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$$

$$= 3.111 , \cdots$$

Then $\sum_{i=1}^{\infty} a_i = 3.111111\cdots = \frac{28}{9}$, which means that the series $\sum_{i=1}^{\infty} a_i$ is convergent and the sum of the series is $\frac{28}{9}$.

Example 15.3: Given the sequence $\left\{\frac{1}{2^n}\right\}$ which has $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2^2} = \frac{1}{4}$, $a_3 = \frac{1}{2^3} = \frac{1}{8}$, $a_4 = \frac{1}{2^4} = \frac{1}{16}$, $\cdots$.

Then the partial sums of the infinite series $\sum_{i=1}^{\infty} a_i$ will be as follows :

$$S_1 = \sum_{i=1}^{1} a_i = a_1 = \frac{1}{2} = 1 - \frac{1}{2} ,$$

$$S_2 = \sum_{i=1}^{2} a_i = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4} = 1 - \frac{1}{2^2} ,$$

$$S_3 = \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$= 1 - \frac{1}{8} = 1 - \frac{1}{2^3}$,

$s_4 = \sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$= 1 - \frac{1}{16} = 1 - \frac{1}{2^4}, \ldots,$ which implies that the nth partial sum is $s_n = \sum_{i=1}^{n} a_i = 1 - \frac{1}{2^n}$.

Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right)$

$= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$.

Therefore the infinite series $\sum_{i=1}^{\infty} a_i$ converges and the sum of the series is 1 (i.e. $\sum_{i=1}^{\infty} a_i = 1$).

Definition 15.4: A series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{i=1}^{\infty} a r^{i-1}$$

is called a geometric series, where a and r are fixed real numbers, and for each $i \geq 1$ the ith term is r times the term before it.

Example 15.5: The series $\sum_{i=1}^{\infty} 3 \cdot 10^{-i}$ $= \sum_{i=1}^{\infty} \frac{3}{10^i}$ is a geometric series. Show that this geometric series converges to $\frac{1}{3}$.

Solution: This geometric series has $a = \frac{3}{10}$ and $r = \frac{1}{10}$.

The partial sums of this series will be as follows:

$$S_1 = \sum_{i=1}^{1} \frac{3}{10^i} = \frac{3}{10} = 0.3 \quad ,$$

$$S_2 = \sum_{i=1}^{2} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{10^2} = 0.3 + 0.03 = 0.33 \quad ,$$

$$S_3 = \sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333 \quad ,$$

$$S_4 = \sum_{i=1}^{4} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} = 0.3333 \quad , \ldots$$

Then $\displaystyle\sum_{i=1}^{\infty} \frac{3}{10^i} = 0.33333\cdots = \frac{1}{3}$.

Therefore the geometric series $\displaystyle\sum_{i=1}^{\infty} \frac{3}{10^i}$ converges to $\frac{1}{3}$.

Example 15.6: Show that the geometric series $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1}$ converges to 2.

Solution: This geometric series has

$a=\left(\frac{1}{2}\right)^{1-1}=\left(\frac{1}{2}\right)^{0}=1$ and $r=\frac{1}{2}$.

The partial sums of this series will be as follows:

$$S_1=\sum_{i=1}^{1}\left(\frac{1}{2}\right)^{i-1}=\left(\frac{1}{2}\right)^{0}=1=2-1,$$

$$S_2=\sum_{i=1}^{2}\left(\frac{1}{2}\right)^{i-1}=\left(\frac{1}{2}\right)^{0}+\left(\frac{1}{2}\right)^{1}=1+\frac{1}{2}=\frac{3}{2}$$

$$=2-\frac{1}{2},$$

$$S_3=\sum_{i=1}^{3}\left(\frac{1}{2}\right)^{i-1}=\left(\frac{1}{2}\right)^{0}+\left(\frac{1}{2}\right)^{1}+\left(\frac{1}{2}\right)^{2}$$

$$=1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}=2-\frac{1}{4}$$

$$S_4=\sum_{i=1}^{4}\left(\frac{1}{2}\right)^{i-1}=\left(\frac{1}{2}\right)^{0}+\left(\frac{1}{2}\right)^{1}+\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{2}\right)^{3}$$

$$=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}=2-\frac{1}{8},\ \cdots,$$

$$S_n=\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{i-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}$$

$$=2-\frac{1}{2^{n-1}}.$$

Therefore $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i-1}=\lim_{n\to\infty}S_n$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{i-1}=\lim_{n\to\infty}\left(2-\frac{1}{2^{n-1}}\right)$$

$$= \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{1}{2^{n-1}} = 2 - 0 = 2.$$

Theorem 15.7 (Geometric Series Theorem):

If $|r| < 1$, the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{i=1}^{\infty} ar^{i-1}$$

converges to $\frac{a}{1-r}$.

If $|r| \geq 1$ and $a \neq 0$, the geometric series diverges.

If $a = 0$, the geometric series converges to 0.

Example 15.8: Find $\sum_{i=1}^{\infty} \frac{1}{3^{i+1}}$ if the geometric series $\sum_{i=1}^{\infty} \frac{1}{3^{i+1}}$ converges.

Solution:

The series $\sum_{i=1}^{\infty} \frac{1}{3^{i+1}} = \sum_{i=1}^{\infty} \left(\frac{1}{3^2}\right)\left(\frac{1}{3^{i-1}}\right)$

$= \sum_{i=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{i-1}$ is a geometric series