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Complex Function Separation

الفصل الثاني

المحاضرة الاولى

تجزئة الدوال العقدية الى اجزائها الحقيقية و الخيالية

Complex function $w = f(z) = U(x,y) + \boldsymbol{i}\, V(x,y)$

Example 1: Separate each of the following functions into real and imaginary parts , *or in other words*

Find :

$U(x,y)$ *and* $V(x,y)$ such that $f(z) = U + \boldsymbol{i}\, V$

(a) $f(z) = z$

(b) $f(z) = z^2 - i$

(c) $f(z) = 2z^2 - 3iz$

(d) $f(z) = z + \frac{1}{z}$

(e) $f(z) = \frac{z^2+1}{z}$

(f) $f(z) = \frac{1-z}{1+z}$

(g) $f(z) = \sqrt{z}$

(h) $f(z) = lnz$

(i) $f(z) = e^{3iz}$

(j) $f(z) = sin2z$

(k) $f(z) = cosz$

(l) $f(z) = z^z$

(m) $f(z) = z^2 e^{2z}$

(n) $f(z) = sinhz$

Solution

(a) $f(z) = z$

$$f(z) = x + iy \rightarrow u(x,y) = x \ , v(x,y) = y$$

(b) $f(z) = z^2 - i$

$$f(z) = (x + iy)^2 - i$$

$$f(z) = x^2 + 2xyi - y^2 - i$$

$$f(z) = (x^2 - y^2) + i(2xy - 1)$$

U(x, y) **V(x, y)**

(c) $f(z) = 2z^2 - 3iz$

$$f(z) = 2(x + iy)^2 - 3i(x + iy)$$

$$f(z) = 2[x^2 + 2xyi - y^2] - 3xi + 3y$$

$$f(z) = 2x^2 + 4xyi - 2y^2 - 3xi + 3y$$

$$f(z) = (2x^2 - 2y^2 - 3y) + i\,(4xy - 3x)$$

U(x, y) **V(x, y)**

(d) $f(z) = z + \frac{1}{z}$ *the same as* (e) $f(z) = \frac{z^2+1}{z}$

Method#1

$$f(z) = (x + iy) + \frac{1}{(x + iy)} = (x + iy) + \frac{1}{(x + iy)} \times \frac{x - iy}{x - iy}$$

$$f(z) = z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2}$$

$$f(z) = x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right)$$

$$f(x + iy) = \left[\frac{x(x^2+y^2)+x}{x^2+y^2}\right] + i\left[\frac{y(x^2+y^2)-y}{x^2+y^2}\right]$$

U(x, y) **V(x, y)**

Method#2

$$f(z) = (x + iy) + \frac{1}{(x + iy)} = \frac{(x + iy)^2 + 1}{(x + iy)} = \frac{x^2 + 2xyi - y^2 + 1}{x + iy}$$

$$z + \frac{1}{z} = \frac{(x^2-y^2+1)+2xyi}{x+iy} \times \frac{x-iy}{x-iy}$$

$$f(z) =$$

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Method#3

This problem can be sovled with the aid of *complex conjugate* of z , i.e.

$f(z) = z + \frac{1}{z}$

$$f(z) = \frac{z^2+1}{z} \quad \rightarrow \quad f(z) = \frac{z^2+1}{z} \times \frac{\bar{z}}{\bar{z}}$$

$$f(z) = \frac{z^2\bar{z} + \bar{z}}{z\bar{z}} = \frac{z^2\bar{z} + \bar{z}}{|z|^2}$$

Then substitute $z = x + iy$ in above equation …what about the result. Disucss!!.

(f) $f(z) = \frac{1-z}{1+z}$

$$f(x+iy) = \frac{1-(x+iy)}{1+(x+iy)} = \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{(1-x)(1+x) - iy(1+x) - iy(1-x) - y^2}{(1+x)^2 + y^2}$$

$$= \frac{1 - x^2 - iy - xyi - iy + xyi - y^2}{(1+x)^2 + y^2}$$

$$= \frac{1 - x^2 - 2iy - y^2}{(1+x)^2 + y^2} = \frac{1 - x^2 - y^2 - 2iy}{(1+x)^2 + y^2}$$

$$f(z) = \underbrace{\frac{1-x^2-y^2}{(1+x)^2+y^2}}_{U(x,y)} - i\underbrace{\frac{2y}{(1+x)^2+y^2}}_{V(x,y)}$$

(g) $f(z) = \sqrt{z}$

$$f(x+iy) = (x+iy)^{1/2}$$

Or in *polar form* , since $z = re^{i\theta} \rightarrow \quad z = r[cos\theta + isin\theta]$

Then ,

$$z^{1/n} = r^{1/n}[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)]$$

$k = 0$, $n = 2$, $r = \sqrt{x^2 + y^2}$, $\theta = tan^{-1}(y/x)$

$$\therefore \sqrt{z} = z^{1/2} = (x^2 + y^2)^{1/2}[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)]$$

Where

$$x = r\cos\theta \quad , \quad \cos\theta = \frac{x}{r}$$

$$y = r\sin\theta \quad , \quad \sin\theta = \frac{y}{r}$$

$$\therefore \sqrt{z} = z^{1/2} = r^{1/2}\left[\frac{x}{2r} + i\frac{y}{2r}\right] = \frac{1}{2\sqrt{r}}[x + iy]$$

$$\therefore z^{1/2} = \frac{1}{2(x^2 + y^2)^{1/2}}[x + iy] = \frac{x}{2(x^2 + y^2)^{1/2}} + i\frac{y}{2(x^2 + y^2)^{1/2}}$$

$\boldsymbol{U(x,y)}$ $\boldsymbol{V(x,y)}$

(h) $f(z) = lnz$

Since , $lnz = lnr + i\theta$

$$U(x,y) = lnr$$

$$V(x,y) = \theta$$

Since

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = tan^{-1}(y/x)$$

$$lnz = ln(x^2 + y^2)^{1/2} + \boldsymbol{i}\, tan^{-1}(y/x)$$

$$lnz = \frac{1}{2} ln((x^2 + y^2) + \boldsymbol{i}\, tan^{-1}(y/x)$$

$\therefore$ $U(x,y) = \frac{1}{2}ln((x^2 + y^2)$ and $V(x,y) = tan^{-1}(y/x)$

(i) $\boldsymbol{f(z) = e^{3iz}}$

$$e^{3iz} = e^{3i(x+iy)} = e^{3ix-3y} = e^{3ix} \times e^{-3y}$$

$$= e^{-3y}[\cos(3x) + \boldsymbol{i}\sin(3x)]$$

$\therefore \quad e^{3iz} = e^{-3y}\cos(3x) + i\, e^{-3y} sin(3x)]$

$\therefore \quad U(x,y) = e^{-3y} cos(3x) \quad \& \quad V(x,y) = e^{-3y} sin(3x)$

(j) $f(z) = sin(2z)$

$f(x+iy) = si\,n\,2(x+iy) = \sin(2x+2iy)$

$= si\,n(\,2x)\cosh(2y) + i\,cos(2x)\;sin(2y)$

$sin(2z) = si\,n(\,2x)\,cosh(2y) + cos(2x)\,sin\,h(2y)$

Where ,

$$sin(iz) = isinh(z)$$

(k) $f(z) = sin(2z)$

Method#1

Let , $f(z) = cosz$

$f(x+iy) = \cos(x+iy)$

باستخدام جيب تمام حاصل جمع زاويتين ، فان

$$\text{Cos}(x+iy) = cosxcos(iy) - sinxsin(iy)$$

ذكرنا سابقا في موضوع الدوال العقدية ، ان هناك علاقة تربط بين الدوال المثلثية (جيب وجيب تمام) بالدوال الزائدية ، اي

$$sin(iy) = isinhy$$
$$cos(iy) = coshy$$

Method#2

$f(z) = cosz$

كما هو معلوم،

$$cosz = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

وبالتعويض عن $z = x + iy$ بالمعادلة اعلاه ، نحصل على ،

$$cosz = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}\left[e^{ix}.e^{-y} + e^{-ix}.e^{y}\right]$$

$$= \frac{1}{2}[e^{-y}(cosx + isinx) + e^{y}(cosx - isinx)]$$

$$= \frac{1}{2}[e^{-y}cosx + ie^{-y}sinx + e^{y}cosx - ie^{y}sinx]$$

الان نجمع الاجزاء الحقيقية مع بعضها والخيالية مع بعضها ، فتصبح ،

$$cosz = \frac{1}{2}[(e^{-y}cosx + e^{y}cosx) + i(e^{-y}sinx - e^{y}sinx)]$$

ناخذ عامل مشترك بين الاجزاء الحقيقية و الاجزاء الخيالية،نحصل على

$$cosz = \frac{1}{2}[cosx(e^{y} + e^{-y}) - isinx\,(e^{y} - e^{-y})]$$

بما ان ،

$$sinhz = \frac{1}{2}(e^{y} - e^{-y})$$

$$coshz = \frac{1}{2}(e^{y} + e^{-y})$$

وعليه فان ، $cosz = cosxcoshy - i\,sinxsinhy$

(I) $f(z) = z^{z}$

Let , $w = z^{z}$

$$lnw = ln(z^{z}) = zln(z)$$

$$w = e^{zlnz}$$

Then ,

$$w = e^{(x+iy)\ln(x+iy)}$$

Since ,

$$ln(z) = lnr + i\theta$$

Or,

$$ln(z) = ln\sqrt{x^2 + y^2} + i\tan^{-1}(y/x)$$

$$w = e^{(x+iy)[\frac{1}{2}ln(x^2+y^2)+i\,tan^{-1}(y/x)]}$$

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$$\therefore \quad w = e^{\frac{1}{2}x\, ln(x^2+y^2) - y\, tan^{-1}(y/x)} \times e^{\,i\, [\frac{y}{2} ln(x^2+y^2) + x\, tan^{-1}(y/x)}$$

(m) $f(z) = z^2 e^{2z}$

$$f(x+iy) = (x+iy)^2 e^{2(x+iy)}$$

$$= (x^2 - y^2 + 2ixy)e^{2x} * \; e^{2yi}$$

$$= e^{2x}[(x^2 - y^2 + 2ixy)] \times [\cos(2y) + i sin(2y)]$$

$$z^2e^{2z} = e^{2x}(x^2 - y^2)\cos(2y) + \boldsymbol{i}\; e^{2x}(x^2 - y^2) sin(2y)]$$

$$+\, \boldsymbol{i}\; e^{2x}(2xy)\cos(2y) - e^{2x}(2xy)\sin(2y)$$

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