Theorem 14.28: If a sequence $\{a_n\}$ converges to L, then all its subsequences converges to L.

Remark 14.29: If any subsequence of a sequence $\{a_n\}$ diverges or if two subsequences have different limits, then $\{a_n\}$ diverges.

Examples 14.30:

1) Since the sequence $\{\frac{1}{n}\}$ converges to 0 which means that $\lim_{n\to\infty} \frac{1}{n} = 0$, then all its subsequences given in example 14.26 converges to 0. That means

$$\lim_{n\to\infty} \frac{1}{n+1} = 0,$$

$$\lim_{n\to\infty} \frac{1}{n+3} = 0,$$

$$\lim_{n\to\infty} \frac{1}{2n} = 0,$$

$$\lim_{n\to\infty} \frac{1}{5n} = 0,$$

$$\lim_{n\to\infty} \frac{1}{2n+1} = 0,$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0,$$

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

2) Since the sequence $\{2n\}$ diverges and $\{2n\}$ is a subsequence of the sequence $\{(-1)^n \cdot n\}$, then the sequence $\{(-1)^n \cdot n\}$ diverges.

3) The sequence $\{(-1)^n\}$ whose terms are $-1, 1, -1, 1, -1, 1, \ldots$ diverges since the sequence $\{(-1)^{2n}\}$ whose terms are $1, 1, 1, 1, 1, \ldots$ is a subsequence of the sequence $\{(-1)^n\}$ and the sequence $\{(-1)^{2n}\}$ converges to 1 (i.e. $\lim_{n\to\infty} (-1)^{2n} = \lim_{n\to\infty} 1 = 1$) and the sequence $\{(-1)^{2n-1}\}$ whose terms are $-1, -1, -1, -1, \ldots$ is a subsequence of the sequence $\{(-1)^n\}$ and the sequence $\{(-1)^{2n-1}\}$ converges to -1 (i.e. $\lim_{n\to\infty} (-1)^{2n-1} = \lim_{n\to\infty} -1 = -1$).

In otherwords the sequence $\{(-1)^n\}$ has two subsequences with different limits, and hence the sequence $\{(-1)^n\}$ diverges.

Theorem 14.31: If the sequence $\{a_n\}$ diverges and if k is any number different from 0, then the sequence $\{k a_n\}$ diverges.

Example 14.32: Since the sequence $\{n\}$

diverges, then the sequence $\{7n\}$ diverges.

Theorem 14.33: If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,

and for some integer N, we have $a_n \leq b_n \leq c_n \quad \forall n \geq N$, then $\lim_{n\to\infty} b_n = L$.

Example 14.34: $\lim_{n\to\infty} \frac{\cos n}{n} = 0$ since

$\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \quad \forall n \geq 1$ and

$\lim_{n\to\infty} \frac{-1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 14.35: $\lim_{n\to\infty} \frac{1}{2^n} = 0$ since

$0 < \frac{1}{2^n} < \frac{1}{n} \quad \forall n \geq 1$ and

$\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 14.36: $\lim_{n\to\infty} \left((-1)^n \cdot \frac{1}{n}\right) = 0$

since $\frac{-1}{n} \leq (-1)^n \cdot \frac{1}{n} \leq \frac{1}{n} \quad \forall n \geq 1$

and $\lim_{n\to\infty} \frac{-1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 14.37: Find $\lim_{n\to\infty} \frac{\sin n}{n}$.

Solution: $-1 \leq \sin n \leq 1 \quad \forall n \geq 1$

$$\Rightarrow \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \forall n \geq 1$$

$$\Rightarrow 0 = \lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Theorem 14.38: Let $\{a_n\}$ be a sequence of real numbers. If $\lim\limits_{n\to\infty} a_n = L$ and if f is a function that is continuous at L and defined at all a_n, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$.

Example 14.39: Find $\lim\limits_{n\to\infty} \sqrt{\frac{n+1}{n}}$.

Solution: $\lim\limits_{n\to\infty} \frac{n+1}{n} = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)$

$$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1+0 = 1.$$

Let $f(x) = \sqrt{x}$, then f is continuous at $x = L = 1$, and defined at all $\frac{n+1}{n}$, hence by theorem 14.38 we have

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \sqrt{1} = 1.$$

Example 14.40: Find $\lim\limits_{n\to\infty} 2^{\frac{1}{n}}$.

Solution: $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Let $f(x) = 2^x$, then f is continuous at $x = L = 0$, and defined at all $\frac{1}{n}$, hence by theorem 14.38 we have

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1.$$

Theorem 14.41: Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for all $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L.$$

Examples 14.42: Find each of the following limits:

1) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$.

2) $\lim\limits_{n\to\infty} \dfrac{n^2+7n}{n^3}$

3) $\lim\limits_{n\to\infty} \dfrac{2^n}{7n}$

4) $\lim\limits_{n\to\infty} \dfrac{7n^3-5n}{2n^3-10}$

Solution:

(by L'Hopital rule)

1) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1} = \dfrac{0}{1} = 0$, and the function $f(x) = \dfrac{\ln x}{x}$ is defined for all $x \geq 1$.

Then $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$.

2) $\lim\limits_{x\to\infty} \dfrac{x^2+7x}{x^3} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+\frac{7}{x^2}}{1} = \dfrac{0}{1} = 0$, and the function $f(x) = \dfrac{x^2+7x}{x^3}$ is defined for all $x \geq 1$. Then $\lim\limits_{n\to\infty} \dfrac{n^2+7n}{n^3} = 0$.

3) $\lim\limits_{x\to\infty} \frac{2^x}{7x} = \lim\limits_{x\to\infty} \frac{2^x \ln 2}{7}$ (by L'Hopital rule)

$=\infty$, and the function $f(x)=\frac{2^x}{7x}$ is defined for all $x \geq 1$. Then $\lim\limits_{n\to\infty} \frac{2^n}{7n} = \infty$ and the sequence $\left\{\frac{2^n}{7n}\right\}$ diverges.

4) $\lim\limits_{x\to\infty} \frac{7x^3-5x}{2x^3-10} = \lim\limits_{x\to\infty} \frac{7-\frac{5}{x^2}}{2-\frac{10}{x^3}} = \frac{7-0}{2-0} = \frac{7}{2}$, and the function $f(x)=\frac{7x^3-5x}{2x^3-10}$ is defined for all $x \geq 1$. Then $\lim\limits_{n\to\infty} \frac{7n^3-5n}{2n^3-10} = \frac{7}{2}$.

Theorem 14.43:

1. $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$,
2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$,
3. $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$ for any $x > 0$,
4. $\lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for any x ,
5. $\lim\limits_{n\to\infty} x^n = 0$ when $|x| < 1$,
6. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$ for any x .

Examples 14.44: Find each of the following limits:

1) $\lim\limits_{n\to\infty} (0.9)^{n+3}$,

2) $\lim\limits_{n\to\infty} \sqrt[n]{n^2}$,

3) $\lim\limits_{n\to\infty} \left(1+\frac{3}{n}\right)^{4n}$,

4) $\lim\limits_{n\to\infty} \frac{40^{2n}}{n!}$,

5) $\lim\limits_{n\to\infty} \frac{3^{n+2}}{(n+1)!}$,

6) $\lim\limits_{n\to\infty} \frac{3n+\ln n^2}{n}$,

7) $\lim\limits_{n\to\infty} \sqrt[n]{15}$.

Solution:

1) $\lim\limits_{n\to\infty} (0.9)^{n+3} = \lim\limits_{n\to\infty} \left((0.9)^3 \cdot (0.9)^n\right)$

$= (0.9)^3 \cdot \lim\limits_{n\to\infty} (0.9)^n$

$= (0.9)^3 \cdot 0$ (by theorem 14.43 (5))

$= 0$.

2) $\lim\limits_{n\to\infty} \sqrt[n]{n^2} = \lim\limits_{n\to\infty} \left(\sqrt[n]{n} \cdot \sqrt[n]{n}\right)$

$= \lim\limits_{n\to\infty} \sqrt[n]{n} \cdot \lim\limits_{n\to\infty} \sqrt[n]{n}$

$= 1 \cdot 1$ (by theorem 14.43 (2))

$= 1$.

3) $\lim\limits_{n\to\infty} \left(1+\frac{3}{n}\right)^{4n} = \left(\lim\limits_{n\to\infty} \left(1+\frac{3}{n}\right)^n\right)^4$

$= \left(e^{3}\right)^{4}$ (by theorem 14.43 (4))

$= e^{12}$.

4) $\lim\limits_{n\to\infty} \dfrac{40^{2n}}{n!} = \lim\limits_{n\to\infty} \dfrac{(40^{2})^{n}}{n!}$

$= \lim\limits_{n\to\infty} \dfrac{(1600)^{n}}{n!} = 0$. (by theorem 14.43(6))

5) $\lim\limits_{n\to\infty} \dfrac{3^{n+2}}{(n+1)!} = \lim\limits_{n\to\infty} \left(\dfrac{3^{2}}{n+1} \cdot \dfrac{3^{n}}{n!}\right)$

$= \lim\limits_{n\to\infty} \dfrac{9}{n+1} \cdot \lim\limits_{n\to\infty} \dfrac{3^{n}}{n!}$

$= 0 \cdot \lim\limits_{n\to\infty} \dfrac{3^{n}}{n!} = 0 \cdot 0$ (by theorem 14.43(6))

$= 0$.

6) $\lim\limits_{n\to\infty} \dfrac{3n + \ln n^{2}}{n} = \lim\limits_{n\to\infty} \left(3 + \dfrac{2\ln n}{n}\right)$

$= \lim\limits_{n\to\infty} 3 + 2 \lim\limits_{n\to\infty} \dfrac{\ln n}{n}$

$= 3 + 2 \lim\limits_{n\to\infty} \dfrac{\ln n}{n}$

$= 3 + 2 \cdot 0$ (by theorem 14.43(1))

$= 3$

7) $\lim\limits_{n\to\infty} \sqrt[n]{15} = \lim\limits_{n\to\infty} 15^{\frac{1}{n}}$

$= 1$. (by theorem 14.43 (3))