nth term is defined by $a_n = \frac{1}{n}$ can be written as $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ where the first term is $a_1 = 1$, the second term is $a_2 = \frac{1}{2}$, the third term is $a_3 = \frac{1}{3}$, $\ldots$, the nth term is $a_n = \frac{1}{n}, \ldots$. The range of this sequence is $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$.

Example 14.6: The sequence $\{a_n\}$ whose nth term is defined by $a_n = 1 - \frac{1}{n}$ can be written as $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1-\frac{1}{n}, \ldots$ where the first term is $a_1 = 0$, the second term is $a_2 = \frac{1}{2}$, the third term is $a_3 = \frac{2}{3}$, the fourth term is $a_4 = \frac{3}{4}, \ldots$, the nth term is $a_n = 1 - \frac{1}{n}, \ldots$. The range of this sequence is $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1-\frac{1}{n}, \ldots\}$.

Example 14.7: The sequence $\{a_n\}$ whose nth term is defined by $a_n = (-1)^{n+1} \cdot \frac{1}{n}$ can be written as $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1} \cdot \frac{1}{n}, \ldots$ where the first term is $a_1 = 1$, the second term is $a_2 = -\frac{1}{2}$, the the third term is $a_3 = \frac{1}{3}$, the fourth term is $a_4 = -\frac{1}{4}, \ldots$, the nth term is $a_n = (-1)^{n+1} \cdot \frac{1}{n}, \ldots$. The range of this sequence is $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1} \cdot \frac{1}{n}, \ldots\}$.

Example 14.8: The sequence $\{a_n\}$ whose nth term is defined by $a_n = \sqrt{n}$ can be written as $1, \sqrt{2}, \sqrt{3}, 2, \ldots, \sqrt{n}, \ldots$

where the first term is $a_1 = 1$, the second term is $a_2 = \sqrt{2}$, the third term is $a_3 = \sqrt{3}$, the fourth term is $a_4 = 2$, ..., the nth term is $a_n = \sqrt{n}$, ... The range of this sequence is $\{1, \sqrt{2}, \sqrt{3}, 2, \ldots, \sqrt{n}, \ldots\}$.

Example 14.9: The sequence $\{a_n\}$ whose nth term is defined by

$$a_n = \begin{cases} \sin nx & \text{if } n \text{ is odd} \\ \cos nx & \text{if } n \text{ is even} \end{cases}$$

can be written as $\sin x$, $\cos 2x$, $\sin 3x$, $\cos 4x$, ..., $\sin nx$ (n is odd), $\cos nx$ (n is even), ... where the first term is $a_1 = \sin x$, the second term is $a_2 = \cos 2x$, the third term is $a_3 = \sin 3x$, the fourth term is $a_4 = \cos 4x$, ..., the nth term (n is odd) is $\sin nx$, the nth term (n is even) is $\cos nx$, ... The range of this sequence is $\{\sin x, \cos 2x, \sin 3x, \cos 4x, \ldots, \sin nx$ (n is odd), $\cos nx$ (n is even), $\ldots\}$.

Example 14.10: The sequence $\{a_n\}$ whose nth term is defined by $a_n = 5$ can be written as $5, 5, 5, 5, \ldots, 5, \ldots$ where the first term is $a_1 = 5$, the second term is $a_2 = 5$, the third term is $a_3 = 5$, the fourth term is $a_4 = 5$, ..., the nth term is $a_n = 5$, ... The range of this sequence is $\{5\}$.

Exercise 14.11: Find the values of the first six terms of each of the following sequences:

1. $\left\{ \frac{1-n}{n^2} \right\}$
2. $\left\{ \frac{2^n}{2!} \right\}$
3. $\left\{ 2n \ln n \right\}$
4. $\left\{ \left(-\frac{1}{n}\right)^n \right\}$
5. $\left\{ \frac{(-1)^n}{2n-1} \right\}$.

Definitions 14.12: The sequence $\{a_n\}$ converges to the number L if to every positive number ϵ there is an integer N such that $|a_n - L| < \epsilon$ for all $n > N$. If no such number L exists, then we say that the sequence $\{a_n\}$ diverges.

If the sequence $\{a_n\}$ converges to L, then we write $\lim_{n \to \infty} a_n = L$ or we write $a_n \to L$, and we call L the limit of the sequence $\{a_n\}$.

Remark 14.13: The sequence $\{a_n\}$ converges to L, if for every positive number ϵ, there is an index N such that all the terms after the Nth term lie within the distance ϵ from L.

Example 14.14: Show that the sequence $\left\{ \frac{1}{n} \right\}$ converges to 0.

Solution: Let $\epsilon > 0$ be any given positive number and let N be the least integer greater than or equal to $\frac{1}{\epsilon}$. Then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} \leq \epsilon \quad \text{for all } n > N$$

since $N \geq \frac{1}{\epsilon}$. This proves that $\{\frac{1}{n}\}$ converge to 0.

Example 14.15: Show that the sequence $\{12\}$ converges to 12.

Solution: Let $\epsilon > 0$ be any positive number. Then

$$|a_n - L| = |12 - 12| \quad (\text{since } a_n = 12 \text{ and } L = 12)$$

$$= 0 < \epsilon \quad \text{for all } n \geq 1.$$

This proves $\{12\}$ converges to 12.

Example 14.16: Show that the sequence $\{\frac{1}{n^2}\}$ converges to 0.

Solution: Let $\epsilon > 0$ be any given positive number and let N be the least positive integer greater than or equal to $\frac{1}{\epsilon}$. Then

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{n} < \frac{1}{N} \leq \epsilon \quad \text{for all } n > N \text{ since } N \geq \frac{1}{\epsilon}.$$

This proves that $\{\frac{1}{n^2}\}$ converges to 0.

Example 14.17: Show that the sequence $\{\frac{1}{n+1}\}$ converges to 0.

Solution: Let $\epsilon > 0$ be any given positive number and let N be the smallest integer greater than or equal $\frac{1}{\epsilon} - 1$. Then $N+1 \geq \frac{1}{\epsilon}$. Thus

$$|a_n - L| = \left|\frac{1}{n+1} - 0\right| \qquad \text{(since } a_n = \frac{1}{n+1} \text{ and } L = 0\text{)}$$

$$= \frac{1}{n+1} < \frac{1}{N+1} \leq \epsilon \quad \text{for all}$$

$n > N$ since $N+1 \geq \frac{1}{\epsilon}$. This proves $\{\frac{1}{n+1}\}$ converges to 0, or $\lim_{n\to\infty} \frac{1}{n+1} = 0$.

Example 14.18: Show that the sequence $\{\frac{n-1}{n}\}$ converges to 1.

Solution: Let $\epsilon > 0$ be any given positive number. Let N be the smallest positive integer such that $N \geq \frac{1}{\epsilon}$. Then

$$\left|\frac{n-1}{n} - 1\right| = \left|\frac{n-1-n}{n}\right| = \left|\frac{-1}{n}\right| = \frac{1}{n} < \frac{1}{N}$$

$$\leq \epsilon \quad \forall n > N \quad \text{since } N \geq \frac{1}{\epsilon}.$$

$$\therefore \left|\frac{n-1}{n} - 1\right| < \epsilon \quad \forall n > N.$$

This proves $\{\frac{n-1}{n}\}$ converges to 1, or

$$\lim_{n\to\infty} \frac{n-1}{n} = 1.$$

Example 14.19: Show that the sequence $\{n+2\}$ diverges.

Solution: Suppose that the sequence $\{n+2\}$ converges to L. Then for $\epsilon = 0.1$ there exists a positive integer N such that $|n+2-L| < \epsilon \ \forall n \geq N$, which implies that

$$|(N+1)+2-L| < \epsilon \text{ and } |(N+10)+2-L| < \epsilon.$$

Thus $0.2 = 2\epsilon = \epsilon + \epsilon > |(N+1)+2-L|$

$+|(N+10)+2-L| = |(N+1)+2-L| +$

$|L-(N+10)-2| \geq |(N+1)+\cancel{2}-\cancel{L}+\cancel{L}-(N+10)-\cancel{2}|$

$= |\cancel{N}+1-\cancel{N}-10| = 9$ C!.

$\therefore$ the sequence $\{n+2\}$ diverges.

Example 14.20: Show that the sequence $\{(n+1)^2\}$ diverges.

Solution: Suppose that the sequence $\{(n+1)^2\}$ converges to L. Then for $\epsilon = 0.1$ there exists a positive integer N such that $|(n+1)^2-L| < \epsilon \ \forall n \geq N$, which implies that

$$|((N+1)+1)^2-L| < \epsilon \text{ and } |((N+10)+1)^2-L| < \epsilon$$

$$\Rightarrow |(N+2)^2-L| < \epsilon \text{ and } |(N+11)^2-L| < \epsilon.$$

Thus $0.2 = 2\epsilon = \epsilon + \epsilon$

$$\geq |(N+2)^2 - L| + |(N+11)^2 - L|$$

$$\geq |(N+2)^2 - \cancel{L} + \cancel{L} - (N+11)^2|$$

$$= |\cancel{N^2} + 4N + 4 - \cancel{N^2} - 22N - 121|$$

$$= |-18N - 117| = 18N + 117 \qquad C!$$

$\therefore$ the sequence $\{(n+1)^2\}$ diverges.

Definition 14.21: The sequence $\{a_n\}$ is said to be bounded if there is a positive number M such that $|a_n| \leq M$ for all positive integers n.

Theorem 14.22: If the sequence $\{a_n\}$ converges, then $\{a_n\}$ is bounded.

Theorem 14.23: If the sequence $\{a_n\}$ converges to both L and L', then $L = L'$ (i.e. if $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} a_n = L'$, then $L = L'$).

Theorem 14.24: If $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$ are both exist and finite,

then

i) $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$,

ii) $\lim\limits_{n\to\infty} (k\, a_n) = kA$ (k is any number),

iii) $\lim_{n\to\infty} (a_n b_n) = AB$,

iv) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ provided that $B \neq 0$ and b_n is never be 0.

Definition 14.25: Let $\{a_n\}$ be a sequence of real numbers and let $\{n_k\}$ be a sequence of positive integers such that $n_1 < n_2 < n_3 < \dots$, then the sequence $\{a_{n_i}\}$ is called a subsequence of $\{a_n\}$.

Example 14.26: Give seven different subsequences of the sequence $\{\frac{1}{n}\}$.

Solution:

The terms of the sequence $\{\frac{1}{n}\}$ are

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots$

1) The sequence $\{\frac{1}{n+1}\}$ whose terms are

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

2) The sequence $\{\frac{1}{n+3}\}$ whose terms are

$\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

3) The sequence $\{\frac{1}{2n}\}$ whose terms are

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

4) The sequence $\{\frac{1}{5n}\}$ whose terms are

$\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

5) The sequence $\{\frac{1}{2n+1}\}$ whose terms are

$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

6) The sequence $\{\frac{1}{n^2}\}$ whose terms are

$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

7) The sequence $\{\frac{1}{2^n}\}$ whose terms are

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$ is a subsequence of $\{\frac{1}{n}\}$.

Example 14.27: Give four different subsequences of the sequence $\{(-1)^n \cdot n\}$.

Solution:

The terms of the sequence $\{(-1)^n \cdot n\}$ are $-1, 2, -3, 4, -5, 6, \dots$

1) The sequence $\{2n\}$ whose terms are $2, 4, 6, 8, 10, \dots$ is a subsequence of $\{(-1)^n \cdot n\}$.

2) The sequence $\{-2n+1\}$ whose terms are $-1, -3, -5, -7, \dots$ is a subsequence of $\{(-1)^n \cdot n\}$.

3) The sequence $\{2n+2\}$ whose terms are $4, 6, 8, 10, 12, \dots$ is a subsequence of $\{(-1)^n \cdot n\}$.

4) The sequence $\{(-1)^n \cdot (n+2)\}$ whose terms are $-3, 4, -5, 6, -7, \dots$ is a subsequence of $\{(-1)^n \cdot n\}$.