§13: Integrals in Polar Coordinates

Let $f(r,\theta)$ be a function defined over a region R that is bounded by the rays $\theta=\alpha$ and $\theta=\beta$ and by the continuous curves $r=g_1(\theta)$ and $r=g_2(\theta)$, and also suppose that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β. Then

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta.$$

1. If $f(r,\theta) \geq 0$ over the region R defined by $\alpha \leq \theta \leq \beta$, $g_1(\theta) \leq r \leq g_2(\theta)$, then

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta = \text{The}$$

volume of the solid which is bounded below by the region R and above by the surface $z=f(r,\theta)$.

2. If $f(r,\theta)=1$, then the area of the region R defined by $\alpha \leq \theta \leq \beta$, $g_1(\theta) \leq r \leq g_2(\theta)$ is

$$\text{The area of } R = \iint_R dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} r\,dr\,d\theta$$

Example 13.1: Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

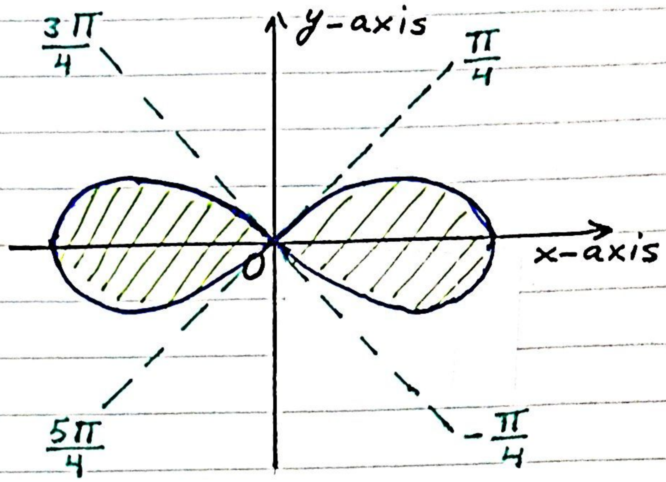
Solution:

To find the limits of the integrals let $\cos 2\theta = 0$

$\Rightarrow 2\theta = \cos^{-1} 0$

$= \mp \frac{\pi}{2}, 2\pi \mp \frac{\pi}{2}$

$= \mp \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$



$\Rightarrow \theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$.

The total area enclosed by the lemniscate curve $r^2 = 4\cos 2\theta$ is 2 times the area of the right side. Thus

The area $A = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cancel{4}^{2}\cos 2\theta}{\cancel{2}_{1}}\, d\theta = 2\left[\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) = 2(1-(-1))$$

$$= 2(2) = 4.$$

Example 13.2: Find the area that is bounded by the curve $r = 2+\cos\theta$ and the rays $\theta = 0$, $\theta = \frac{\pi}{2}$.

Solution:

$$\text{The area} = \int_0^{\frac{\pi}{2}} \int_0^{2+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^{2+\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(2+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (4+4\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(4+4\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(4+4\cos\theta+\frac{1}{2}+\frac{2\cos 2\theta}{4}\right) d\theta$$

$$= \frac{1}{2}\left[4\theta+4\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(2\pi+4+\frac{\pi}{4}+0\right)-0\right]$$

$$= \pi+2+\frac{\pi}{8} = \frac{9\pi}{8}+2.$$

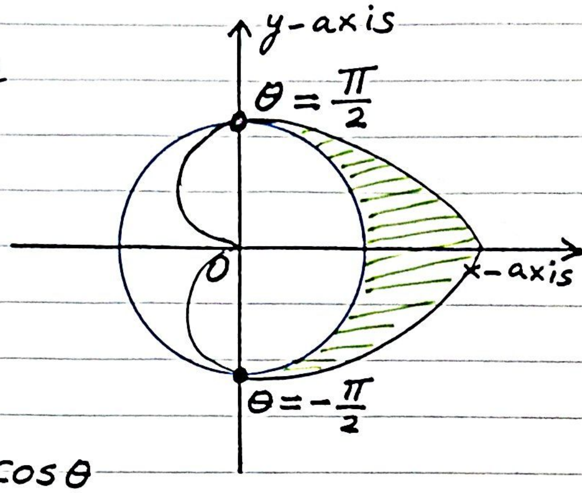
Example 13.3: Find the area that lies inside the cardioid curve $r = 1+\cos\theta$ and outside the circle $r=1$.

Solution:

To find the limits of the integrals let us solve the two equation $r=1$ and $r=1+\cos\theta$, we get that $1=1+\cos\theta$

$\Rightarrow \cos\theta=0$

$\Rightarrow \theta=\cos^{-1}0=\pm\frac{\pi}{2}$



$$\therefore \text{The area } A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{1}^{1+\cos\theta} r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_1^{1+\cos\theta}d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left((1+\cos\theta)^2-(1^2)\right)d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cancel{1}+2\cos\theta+\cos^2\theta-\cancel{1})\,d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2\cos\theta+\frac{1}{2}+\frac{2\cos 2\theta}{4}\right)d\theta$$

$$=\frac{1}{2}\left[2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(2+\frac{\pi}{4}+0\right)-\left(-2-\frac{\pi}{4}+0\right)\right]$$

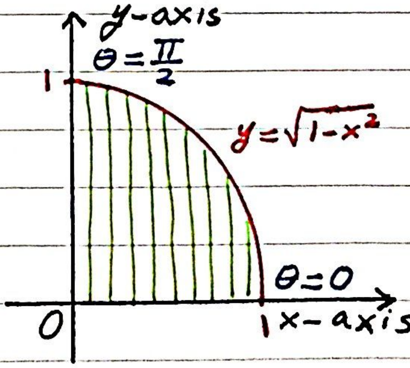
$$= \frac{1}{2}\left(4+\frac{\pi}{2}\right) = 2+\frac{\pi}{4}.$$

Changing Cartesian Integrals to Polar

Integrals 13.4: The general rule for converting a cartesian integral to polar integral is replacing x by $r\cos\theta$, y by $r\sin\theta$, and $dy\,dx$ by $r\,dr\,d\theta$ and then changing the limits of the integral to the polar limits.

Example 13.5: Change the cartesian integral $\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ to the equivalent polar integral and evaluate the polar integral.

Solution:



When $y=0$ and $x=0$ then $r^2=x^2+y^2=0 \Rightarrow r=0$, and when

$y=\sqrt{1-x^2}$ then $y^2=1-x^2$ $\Rightarrow y^2+x^2=1 \Rightarrow r^2=1 \Rightarrow r=1$.
Thus the limits of r is $0\le r\le 1$.

Now if $x=0$ then $y=\sqrt{1-x^2}=1 \Rightarrow r\sin\theta=1$ and $r=1 \Rightarrow \sin\theta=1 \Rightarrow \theta=\sin^{-1}1=\frac{\pi}{2}$, and if $x=1$ then $y=\sqrt{1-x^2}=0 \Rightarrow r\sin\theta=0$ and $r=1 \Rightarrow \sin\theta=0 \Rightarrow \theta=\sin^{-1}0=0$.

Thus the limits of θ is $0 \le \theta \le \frac{\pi}{2}$.

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^3\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\theta$$

$$= \left[\frac{\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}.$$

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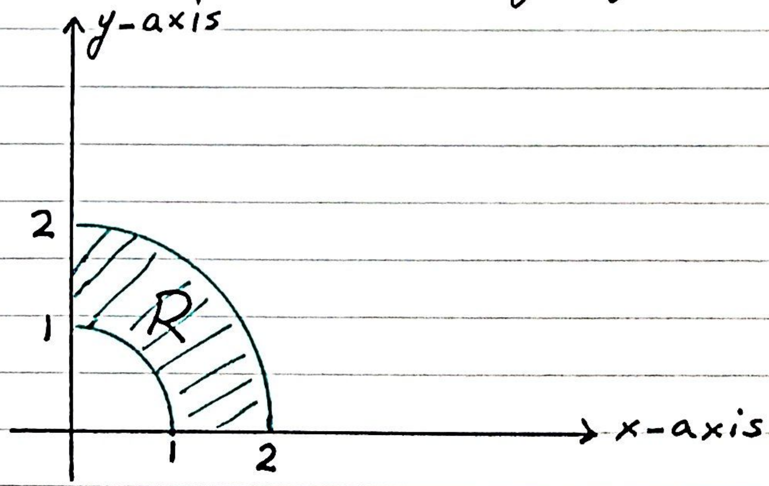
$$\iint_R \frac{1}{x^2+y^2}\, dy\, dx$$, where R is the

region in the first quadrant bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$, where R is shown in the following figure



Solution: Change the cartesian integral

$$\iint_R \frac{1}{x^2+y^2}\, dy\, dx$$ to the polar

integral with r limits is $1 \leq r \leq 2$ and θ limits is $0 \leq \theta \leq \frac{\pi}{2}$ as follows:

$$\iint_R \frac{1}{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{r}\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\ln r\right]_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\ln 2 - \ln 1)\, d\theta = \int_0^{\frac{\pi}{2}} \ln 2\, d\theta$$

$$= \left[(\ln 2).\theta\right]_0^{\frac{\pi}{2}} = (\ln 2).\frac{\pi}{2} - (\ln 2).0$$

$$= \frac{\pi}{2} \ln 2.$$

§14: Sequences of Numbers

Definition 14.1: A sequence of numbers is a function whose domain is the set of positive integers

Definition 14.2: The numbers in the range of a sequence a are called the terms of the sequence a. The number $a(n)$ (which is denoted by a_n) is called the nth term or the term with index n.

Remarks 14.3:

1. The set $\{a_1, a_2, \dots, a_n, \dots\}$ is called the range of the sequence a, and the sequence will be denoted by $\{a_n\}$ or $\langle a_n \rangle$.
2. If the set $\{a_1, a_2, \dots, a_n, \dots\}$ is a subset of the set of all real numbers $\mathbb{R}$, then the sequence $\{a_n\}$ is called a sequence of real numbers.

Example 14.4: The sequence $\{a_n\}$ whose nth term is defined by $a_n = 2n-3$ can be written as $-1, 1, 3, 5, \dots, 2n-3, \dots$ where the first term is $a_1 = -1$, the second term is $a_2 = 1$, the third term is $a_3 = 3$, the fourth term is $a_4 = 5$, $\dots$, and the nth term is $a_n = 2n-3, \dots$. The range of this sequence is $\{-1, 1, 3, 5, \dots, 2n-3, \dots\}$.

Example 14.5: The sequence $\{a_n\}$ whose