§12: Polar Coordinates

To form a polar coordinate system in a plane, start with a fixed point O which is called the pole or the origin, and a fixed half line Ox drawn from the pole O which is called the polar axis or the initial ray, usually taken to be horizontal and its direction is to the right.

If P is any point in the plane, then we associate polar coordinates (r, θ) for it as follows: Starting with the polar axis Ox as the initial side of the angle θ and rotate the terminal side around the pole O until it or its extension (the extension of the terminal side of θ is the half line drawn from the pole O in the opposite direction of the terminal side) passes through the point P.

The angle θ can be measured in radiants or in degrees.

The angle θ is positive if the rotation of the terminal side is counterclockwise and negative if the rotation is clockwise.

r is the directed distance from the pole O to the point P.

r is positive if P is on the terminal side and negative if P is on the extension of the terminal side.

Remark 12.1: Any point P in polar coordinate system with coordinates (r, θ) has also an infinite number of polar coordinates which are:

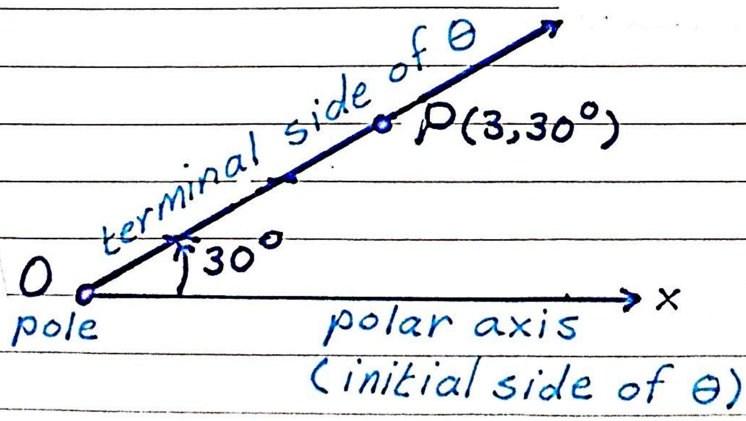
1. a- $(r, \theta + n.180°)$ for all even integers n, when θ is measured in degrees.
   b- $(r, \theta + n\pi)$ for all even integers n, when θ is measured in radiants.
2. a- $(-r, \theta + n.180°)$ for all odd integers n, when θ is measured in degrees.
   b- $(-r, \theta + n\pi)$ for all odd integers n, when θ is measured in radiants.

Example 12.2: Find all the polar coordinates (r, θ) of the point $P(3, 30°)$ such that $-360° \leq \theta \leq 360°$ and plot the point P for all these coordinates.

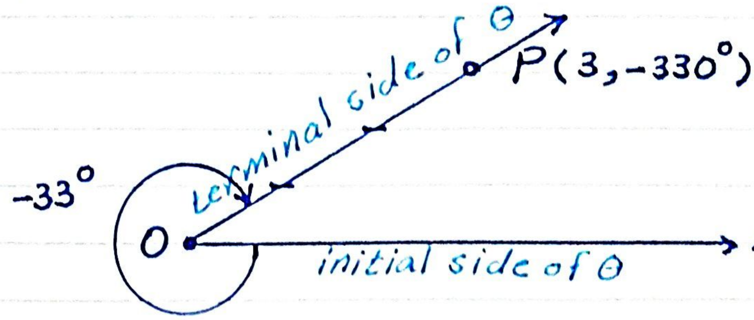
Solution:

The point P has the coordinates $(3, 30°)$, $(3, -330°)$, $(-3, 210°)$, $(-3, -150°)$. i.e.
$P(3, 30°) = P(3, -330°) = P(-3, 210°)$
$= P(-3, -150°)$.

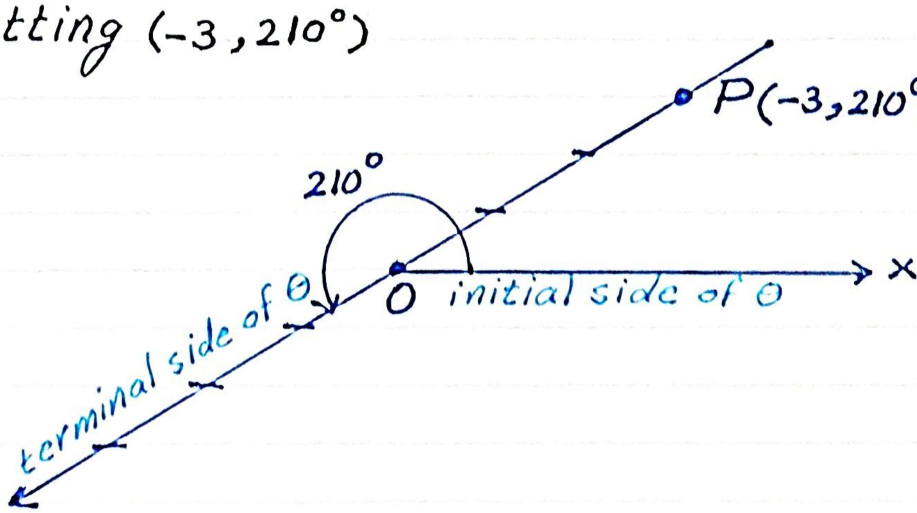
1. Plotting $P(3, 30°)$



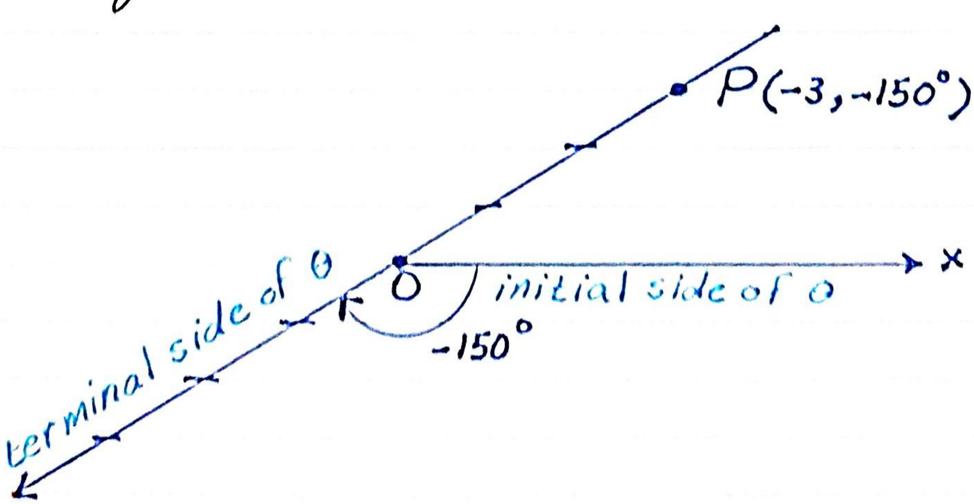
2. Plotting (3, -330°)



3. Plotting (-3, 210°)



4. Plotting (-3, -150°)



Exercise 12.3: Find all the polar coordinates (r,θ) of each of the following points P such that $-360^\circ \leq \theta \leq 360^\circ$ and plot P for each coordinates that you find of P.

1. $P(5, 45^\circ)$
2. $P(4, 90^\circ)$
3. $P(6, 270^\circ)$
4. $P(2, -30^\circ)$
5. $P(-3, 45^\circ)$
6. $P(-2, -30^\circ)$.

Example 12.4: Find all the polar coordinates of the point $P(2, 30^\circ)$.

Solution:

The polar coordinates are:

$(2, 30^\circ + n.180^\circ)$ where $n = 0, \pm 2, \pm 4, \pm 6, \ldots$

and $(-2, 30^\circ + n.180^\circ)$ where $n = \pm 1, \pm 3, \pm 5, \pm 7, \ldots$

Remark 12.5: The polar coordinates of the point $P(2, 30^\circ)$ in example 12.4 can be written as follows:

$(2, 30^\circ), (2, 390^\circ), (2, 750^\circ), (2, 1110^\circ), \ldots$;

$(2, -330^\circ), (2, -690^\circ), (2, -1050^\circ), \ldots$;

$(-2, 210^\circ), (-2, 570^\circ), (-2, 930^\circ), \ldots$;

$(-2, -150^\circ), (-2, -510^\circ), (-2, -870^\circ), \ldots$.

Example 12.6: Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.

Solution:

The polar coordinates are:

$(2, \frac{\pi}{6} + n\pi)$ where $n = 0, \pm 2, \pm 4, \pm 6, \ldots$

and $(-2, \frac{\pi}{6} + n\pi)$ where $n = \pm 1, \pm 3, \pm 5, \ldots$

Remark 12.7: The polar coordinates of the point $P(2, \frac{\pi}{6})$ in example 12.6 can be written as follows:

$(2, \frac{\pi}{6}), (2, \frac{13\pi}{6}), (2, \frac{25\pi}{6}), \ldots$;

$(2, -\frac{11\pi}{6}), (2, -\frac{23\pi}{6}), (2, -\frac{35\pi}{6}), \ldots$;

$(-2, \frac{7\pi}{6}), (-2, \frac{19\pi}{6}), (-2, \frac{31\pi}{6}), \ldots$;

$(-2, -\frac{5\pi}{6}), (-2, -\frac{17\pi}{6}), (-2, -\frac{29\pi}{6}), \ldots$.

Converting Polar Coordinates to Cartesian

Coordinates 12.8:

To convert the polar coordinates to the cartesian coordinate use the following equations

$x = r\cos\theta$ and $y = r\sin\theta$

Example 12.9: Find the cartesian coordinates of the following points:

1. $P(2, 60°)$
2. $P(2, 90°)$
3. $P(-2, 0°)$
4. $P(-2, 90°)$
5. $P(2, \pi/4)$
6. $P(3, 30°)$
7. $P(-6, \pi/3)$.

Solution:

1. For $P(2, 60°)$ we have

$x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow x = 2\cos 60°$ and $y = 2\sin 60°$

$\Rightarrow x = 2\left(\frac{1}{2}\right)$ and $y = 2\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow x = 1$ and $y = \sqrt{3}$

$\Rightarrow P(x, y) = P(1, \sqrt{3})$.

2. For $P(2, 90°)$ we have

$x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow x = 2\cos 90°$ and $y = 2\sin 90°$

$\Rightarrow x = 2(0)$ and $y = 2(1)$

$\Rightarrow x = 0$ and $y = 2$

$\Rightarrow P(x, y) = P(0, 2)$.

3. For $P(-2, 0°)$ we have

$x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow x = -2\cos 0°$ and $y = -2\sin 0°$

$\Rightarrow x = -2(1)$ and $y = -2(0)$

$\Rightarrow x = -2$ and $y = 0$

$\Rightarrow P(x, y) = P(-2, 0)$.

4. For $P(-2, 90^\circ)$ we have
$x = r\cos\theta$ and $y = r\sin\theta$
$\Rightarrow x = -2\cos 90^\circ$ and $y = -2\sin 90^\circ$
$\Rightarrow x = -2(0)$ and $y = -2(1)$
$\Rightarrow x = 0$ and $y = -2$
$\Rightarrow P(x,y) = P(0,-2)$.

5. For $P(2, \frac{\pi}{4})$ we have
$x = r\cos\theta$ and $y = r\sin\theta$
$\Rightarrow x = 2\cos\frac{\pi}{4}$ and $y = 2\sin\frac{\pi}{4}$

$\Rightarrow x = 2\left(\frac{1}{\sqrt{2}}\right)$ and $y = 2\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow x = \sqrt{2}$ and $y = \sqrt{2}$
$\Rightarrow P(x,y) = P(\sqrt{2}, \sqrt{2})$.

5. Exercise.
6. Exercise.

Example 12.10: Find the cartesian equivalent to each of the following polar equations:
1. $r^2\cos^2\theta - r^2\sin^2\theta = 1$,
2. $r^2\cos\theta\sin\theta = 4$,
3. $r\cos\theta = 2$,

Solution:
1. $r^2\cos^2\theta - r^2\sin^2\theta = 1$
$\Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = 1$
$\Rightarrow x^2 - y^2 = 1$.

2. $r^2\cos\theta\sin\theta = 4 \Rightarrow (r\cos\theta)(r\sin\theta) = 4$
$\Rightarrow xy = 4$.

3. $r\cos\theta = 2 \Rightarrow x = 2$.

Example 12.11: Find the cartesian equivalent to each of the following equations and identify their graphs:

1. $r\sin\theta = 5$.
2. $7r\sin\theta + 3r\cos\theta = 12$.
3. $r^2 = 4r\cos\theta$.

Solution:

1. $r\sin\theta = 5 \Rightarrow y = 5$ whose graph is the horizontal line parallel to the x-axis passing through the point $(0,5)$.

2. $7r\sin\theta + 3r\cos\theta = 12 \Rightarrow 7y + 3x = 12$
$\Rightarrow 7y = -3x + 12 \Rightarrow y = -\frac{3}{7}x + \frac{12}{7}$ whose graph is the straight line with slope $-\frac{3}{7}$ and y-intercept $\frac{12}{7}$.

3. $r^2 = 4r\cos\theta$
$\Rightarrow x^2 + y^2 = 4x$
$\Rightarrow x^2 - 4x + y^2 = 0$
$\Rightarrow x^2 - 4x + 4 + y^2 = 4$
$\Rightarrow (x-2)^2 + y^2 = 4$ whose graph is a circle of center $C(2,0)$ and radius $r = 2$.

Exercise 12.12: Find the cartesian equivalent to each of the following equations and identify their graphs:

1. $r = \dfrac{5}{3\cos\theta - 2\sin\theta}$.
2. $r^2 = 6r\sin\theta$.