§11 : Triple Integrals

Triple Integrals over a Closed Bounded Region

Region 11.1: If $f(x,y,z)$ is a function defined on a closed bounded region D in the xyz-space, then the triple integral of f over D can be defined in the following way. We partition a rectangular region E containing D into rectangular cells by planes parallel to the xy-plane and planes parallel to the xz-plane, and planes parallel the yz-plane. We number the cells of volume $\Delta V = \Delta x . \Delta y . \Delta z$ that lies entirely within the region D in some order $\Delta V_1, \Delta V_2, \dots, \Delta V_n$, and choose a point (x_k, y_k, z_k) in each rectangular cell of volume ΔV_k, and form the following sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) . \Delta V_k .$$

If f is continuous on the bounded region D, then as Δx_k, Δy_k, and Δz_k approach zero, the sum S_n will approach a limit

$$\lim_{n \to \infty} S_n = \iiint_D f(x,y,z)\, dV .$$

We call this limit the triple integral of f over D, and this triple integral with respect to dx, dy, and dz will be

$$\iiint_D f(x,y,z)\,dV$$

$$= \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)\,dz\,dy\,dx \,.$$

Properties of the Triple Integrals 11.2 :

1) $$\iiint_D k\,f(x,y,z)\,dV = k\iiint_D f(x,y,z)\,dV$$

(k is any number).

2) $$\iiint_D \big(f(x,y,z)+g(x,y,z)\big)\,dV$$

$$= \iiint_D f(x,y,z)\,dV + \iiint_D g(x,y,z)\,dV \,.$$

3) $$\iiint_D \big(f(x,y,z)-g(x,y,z)\big)\,dV$$

$$= \iiint_D f(x,y,z)\,dV - \iiint_D g(x,y,z)\,dV \,.$$

4) $\iiint_D f(x,y,z)\,dV \geq 0$ if $f(x,y,z) \geq 0$ on D.

5) $$\iiint_D f(x,y,z)\,dV \geq \iiint_D g(x,y,z)\,dV$$

if $f(x,y,z) \geq g(x,y,z)$ on D.

6) $$\iiint_D f(x,y,z)\,dV = \iiint_{D_1} f(x,y,z)\,dV$$

$$+ \iiint_{D_2} f(x,y,z)\,dV + \cdots + \iiint_{D_n} f(x,y,z)\,dV,$$

if D is the union of n nonoverlapping cells $D_1, D_2, \ldots, D_n$.

Example 11.3:

Evaluate $\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\,dz\,dy\,dx$.

Solution: $\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\,dz\,dy\,dx$

$$= \int_{-2}^{5}\int_{0}^{3x} \left[4z\right]_{z=y}^{z=x+2} dy\,dx$$

$$= \int_{-2}^{5}\int_{0}^{3x} (4x+8-4y)\,dy\,dx$$

$$= \int_{-2}^{5} \left[4xy + 8y - 2y^2\right]_{y=0}^{y=3x} dx$$

$$= \int_{-2}^{5} (12x^2 + 24x - 18x^2)\,dx$$

$$= \int_{-2}^{5} (24x - 6x^2)\,dx$$

$$= 12x^2 - 2x^3\Big]_{-2}^{5} = (12(25) - 2(125)) - (12(4) - 2(-8))$$

$$= 50 - 64 = -14.$$

Example 11.4 :

Evaluate $\displaystyle\int_{z=0}^{z=1}\int_{y=0}^{y=z}\int_{x=0}^{x=2} xyz\,dx\,dy\,dz.$

Solution: $\displaystyle\int_{0}^{1}\int_{0}^{z}\int_{0}^{2} xyz\,dx\,dy\,dz$

$$= \int_{0}^{1}\int_{0}^{z} \left[\tfrac{1}{2}x^2yz\right]_{x=0}^{x=2} dy\,dz$$

$$= \int_{0}^{1}\int_{0}^{z} 2yz\,dy\,dz = \int_{0}^{1} \left[y^2z\right]_{y=0}^{y=z} dz$$

$$= \int_{0}^{1} z^3\,dz = \frac{z^4}{4}\Big]_{0}^{1} = \frac{1}{4}.$$

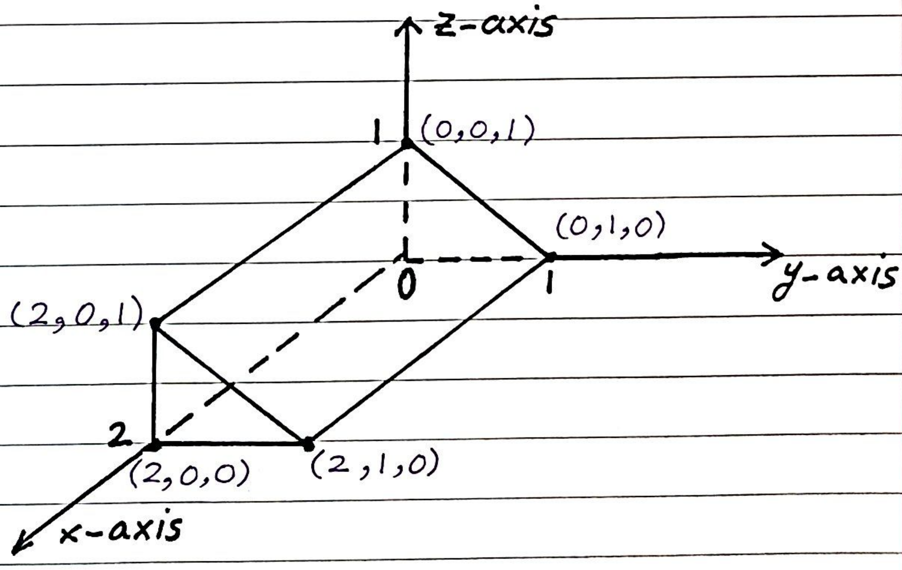
Triple integrals as volumes 11.5 :

If we take $f(x,y,z)=1$ in the triple integrals the triple integral

$$\iiint\limits_{D} f(x,y,z)\,dV = \iiint\limits_{D} dV = \text{the volume}$$

of D.

Example 11.6 : Find the volume V of the solid D which is bounded by the xy-plane, the xz-plane, the yz-plane, the plane $x=2$, and the plane $y+z=1$ as it is shown in the following figure



Solution: $V = \int_0^2 \int_0^1 \int_0^{1-y} dz\,dy\,dx$

$$= \int_0^2 \int_0^1 \left[z \right]_{z=0}^{z=1-y} dy\,dx$$

$$= \int_0^2 \int_0^1 (1-y)\, dy\, dx$$

$$= \int_0^2 \left[y - \frac{y^2}{2} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left(1 - \frac{1}{2}\right) dx$$

$$= \int_0^2 \frac{1}{2}\, dx$$

$$= \frac{x}{2} \Big]_0^2 = 1 .$$

Remark 11.7: The volume V of the solid D in Example 11.6 can be found also by each of the following triple integrals:

1) $V = \int_0^1 \int_0^{1-z} \int_0^2 dx\, dy\, dz$

2) $V = \int_0^1 \int_0^2 \int_0^{1-z} dy\, dx\, dz$

3) $V = \int_0^1 \int_0^{1-y} \int_0^2 dx\, dz\, dy$

4) $V = \int_0^2 \int_0^1 \int_0^{1-z} dy\, dz\, dx$

5) $V = \int_0^1 \int_0^2 \int_0^{1-y} dz\, dx\, dy$.

Example 11.8: Find the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x=1$, $y=2$, and $z=3$.

Solution: $V=\int_0^2\int_0^1\int_0^3 dz\,dx\,dy$

$$=\int_0^2\int_0^1 [z]_0^3\,dx\,dy=\int_0^2\int_0^1 3\,dx\,dy=\int_0^2 [3x]_0^1\,dy$$

$$=\int_0^2 3\,dy=3y\Big]_0^2=6.$$

We can also find V as follows:

$$V=\int_0^3\int_0^2\int_0^1 dx\,dy\,dz=\int_0^3\int_0^2 [x]_0^1\,dy\,dz$$

$$=\int_0^3\int_0^2 dy\,dz=\int_0^3 [y]_0^2\,dz=\int_0^3 2\,dz=2z\Big]_0^3=6.$$

or $V=\int_0^1\int_0^2\int_0^3 dz\,dy\,dx=\int_0^1\int_0^2 [z]_0^3\,dy\,dx$

$$=\int_0^1\int_0^2 3\,dy\,dx=\int_0^1 [3y]_0^2\,dx=\int_0^1 6\,dx=6x\Big]_0^1=6.$$

or $V=\int_0^3\int_0^1\int_0^2 dy\,dx\,dz=\int_0^3\int_0^1 [y]_0^2\,dx\,dz$

$$= \int_0^3 \int_0^1 2\, dx\, dz = \int_0^3 \left[2x\right]_0^1 dz = \int_0^3 2\, dz = 2z\Big]_0^3 = 6.$$

Example 11.9: Find the volume of the solid in the first octant enclosed by the cylinder $x^2+z^2=4$ and the plane $y=3$.

Solution: $V = \int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dz\, dx\, dy$

$$= \int_0^3 \int_0^2 \left[z\right]_0^{\sqrt{4-x^2}} dx\, dy$$

$$= \int_0^3 \int_0^2 \sqrt{4-x^2}\, dx\, dy$$

let $x = 2\sin\theta \Rightarrow$
$\frac{dx}{d\theta} = 2\cos\theta$ and $\sin\theta = \frac{x}{2}$
$\Rightarrow dx = 2\cos\theta\, d\theta$ and
$\theta = \sin^{-1}\frac{x}{2}$
$\Rightarrow 4 - x^2 = 4 - 4\sin^2\theta$
$= 4(1-\sin^2\theta) = 4\cos^2\theta$
and $\theta = 0$ when $x = 0$,
$\theta = \frac{\pi}{2}$ when $x = 2$.

$$= \int_0^3 \int_0^{\frac{\pi}{2}} (2\cos\theta)(2\cos\theta)\, d\theta\, dy$$

$$= \int_0^3 \int_0^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta\, dy = \int_0^3 \int_0^{\frac{\pi}{2}} 4\left(\frac{1+\cos 2\theta}{2}\right) d\theta\, dy$$

$$= \int_0^3 \int_0^{\frac{\pi}{2}} (2 + 2\cos 2\theta)\, d\theta\, dy = \int_0^3 \left[2\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}} dy$$

$$= \int_0^3 (\pi + \sin\pi)\, dy = \int_0^3 \pi\, dy = \pi y\Big]_0^3 = 3\pi.$$