# *Chapter One*

## $S_1$- Matrices المصفوفات

### (1) Matrix

**Def ;-** The ***matrix*** is any rectangular array of real or complex number , which has m rows and n columns , which we can be written of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ . & \\ . & \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{pmatrix}_{m*n}$$

Or $A = (a_{ij})_{m*n}$ , for each $i = 1 , \ldots\ldots, m$ , $j = 1 , \ldots\ldots\ldots, n$

**Def :-**The number of rows and columns is called *the **size*** , or ***dimension*** of matrix and denoted by ( m * n ) and read it as follows ( m by n ) matrix )

**Ex :-**

$$A = \begin{pmatrix} 1 & -6 & 1/8 \\ -5 & 1/2 & 0 \\ 9 & 0 & -4 \end{pmatrix}_{3*3} , \quad B = \begin{pmatrix} -1 & 8 & 9/4 & 0 \\ 11 & 4/7 & -13 & 2 \\ 1 & -12 & 0 & 9 \end{pmatrix}_{3*4}$$

## (2) Operations of Matrices

### *(a ) Equality of Matrices* تساوي المصفوفات

**Def :-**The two matrices are ***equal*** if they have the same dimension and their corresponding elements are equal

**Ex:**

$$A = \begin{pmatrix} -4 & 2 & 5 & 8 & 0 \\ 1/4 & 9 & 0 & -1 & 5 \end{pmatrix}_{2*5} , \quad B = \begin{pmatrix} -4 & 2 & 5 & 8 & 0 \\ 1/4 & 9 & 0 & -1 & 5 \end{pmatrix}_{2*5}$$

**Ex2 :-**

Let $H = \begin{pmatrix} -5 & 1 & 1/7 \\ 8 & 16 & 0 \end{pmatrix}_{2*3}$ , $D = \begin{pmatrix} -5 & 1 & 1/7 \\ -8 & 16 & 0 \end{pmatrix}_{2*3}$

Then A = B , but $H \neq D$ since $8 = h_{21} \neq d_{21} = -8$

## *(b ) Sum and Subtraction of matrices* جمع وطرح المصفوفات

**Def :-** 1- Let A and B be two matrices of the same order then the sum of A and B denoted by A + B , can be found by sum the corresponding elements of A and B

In other words , if $A = (a_{ij})_{m*n}$ , $B = (b_{ij})_{m*n}$

Then **$A + B = (a_{ij})_{m*n} + (b_{ij})_{m*n}$**

$$= \left[ (a_{ij} + b_{ij}) \right]_{m*n}$$

2- Also we can defined the subtraction of matrices with same conditions of the Sum as follows:

**$A - B = (a_{ij})_{m*n} - (b_{ij})_{m*n}$**

$$= \left[ (a_{ij} - b_{ij}) \right]_{m*n}$$

**Ex :-**

Let $A = \begin{pmatrix} -5 & 9 & 0 \\ 6 & -4 & 13 \end{pmatrix}_{2*3}$ , $B = \begin{pmatrix} 15 & 4 & 0 \\ 1 & -3 & -5 \end{pmatrix}_{2*3}$

Find A + B and A - B

**Sol:-**

$$A + B = \begin{pmatrix} -5+15 & 9+4 & 0+0 \\ 6+1 & -4+(-3) & 13+(-5) \end{pmatrix} = \begin{pmatrix} 10 & 13 & 0 \\ 7 & -7 & 8 \end{pmatrix}_{2*3}$$

$$A - B = \begin{pmatrix} -20 & 5 & 0 \\ 5 & -1 & 18 \end{pmatrix}_{2*3}$$

## *(C ) Multiplication by scalar number* الضرب بعدد

**Def :-** The product of a matrix $A_{m*n}$ ***by scalar number k*** denoted by $(kA)_{m*n}$ , is a matrix founded by multiplying each elements of A by k .

By other words, $kA=k\,(a_{ij})_{m*n} =(k\,a_{ij})_{m*n}$ , for each $i = 1 , \ldots\ldots, m$ , $j = 1 , \ldots\ldots\ldots, n$

**Ex :-** Let

$$A = \begin{pmatrix} 3 & 1 & 1/8 \\ 5 & 0 & -2 \\ -5 & 1/7 & 6 \end{pmatrix}_{3*3}$$ , k = 4 find k. A

**Sol :-**

$$K\,A = \begin{pmatrix} 12 & 4 & 1/2 \\ 20 & 0 & -8 \\ -20 & 4/7 & 24 \end{pmatrix}_{3*3}$$

**Exc :-**

$$A = \begin{pmatrix} 6 & -2 & 1/7 \\ -5 & 1/3 & 0 \\ 8 & 11 & -0 \end{pmatrix}_{3*3} \quad , \quad B = \begin{pmatrix} -2 & 3 & 6/5 & 0 \\ 0 & 1 & -10 & 9 \\ 1 & -8 & 0 & 0 \end{pmatrix}_{3*4}$$

k =-1/5 then find A –B , A+ k B , k( B –A )

## ( d ) Matrices product ضرب المصفوفات

**Def :-**The product of two matrices A and B is defined only on the assumption that the number of columns in A is equal to the number of rows in B

.Now ; if $A=(a_{ij})_{m*p}$ and $B=(b_{ij})_{p*n}$

then $(A B)= \sum_{k=1}^{p} a_{ik} . b_{kj} = (c_{ij})_{m*n} = C$ .

**Ex :-**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3*3} \quad , \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3*2}$$

Find A*B

**Sol :-**

$$AB = \begin{pmatrix} a_{11} b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \end{pmatrix}_{3*2}$$

**Ex :-** Let

$$A = \begin{pmatrix} 2 & 0 & -3 \\ -1 & 0 & 4 \end{pmatrix}_{2*3} \quad , \quad B = \begin{pmatrix} -3 & 2 \\ 7 & 3 \\ 0 & -1 \end{pmatrix}_{3*2}$$

find A*B

Sol :-

$$A*B = \begin{pmatrix} (2)(-3)+(0)(7)+(-3)(0) & (2)(2)+(0)(3)+(-3)(-1) \\ (-1)(-3)+(0)(7)+(4)(0) & (-1)(2)+(0)(3)+(4)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 7 \\ 3 & -6 \end{pmatrix}_{2*2}$$

**Exc:-** (H .W)

Let

$$A = \begin{pmatrix} 5 & -4 & 0 \\ 0 & 6 & 1 \\ 8 & 3 & 0 \end{pmatrix}_{3*3} , B = \begin{pmatrix} 8 & 0 & -1 \\ 5 & -1 & 4 \\ 2 & 9 & 0 \end{pmatrix}_{3*3} , C = \begin{pmatrix} 4 & 1 \\ 0 & -8 \\ 3 & 0 \end{pmatrix}_{3*2}$$

**Find** (AB),(BC),(CA),(A+B),(AC)+C ,9(AC) ,12(AC)-4(BC)

**Theorm** **(1)**

**if A , B , C be matrices and K real number then**

**(1 ) A + ( B + C ) = ( A + B ) + C**

**( 2 ) A + B = B + A**

**(3 ) K ( A + B ) = K A + K B**

**( 4 ) A ( B C ) = ( A B ) C**

**( 5) A ( B + C ) = A B + A C**

**Proof :-**

(1) Let A= (aij)m*n , B= (bij)m*n , C =(cij)m*n

**L .H .S**

A+{B+C}=(aij)m*n + {(bij)m*n + (cij)m*n}

= (aij)m*n + { ( bij + cij )m*n } ( by Sum law )

= {( aij +bij +cij )m*n } (by Sum law )

since aij ,bij ,cij are elements in R then they have associative property

= { (aij + bij ) + cij}m*n

= { (aij +bij )m*n +(cij)m*n }

= { (aij)m*n + (bij)m*n } + (cij)m*n

= { A + B } + C

**Remarks :-**

**(1 ) matrices multiplication is not commutative**

**A B ≠ B A**

**Ex :-** Let

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 10 \\ -1 & 1 \end{pmatrix} , \quad B\,A = \begin{pmatrix} 3 & 3 \\ -4 & 0 \end{pmatrix}$$

**( 2) A B may be equal to 0 with neither A nor B equal to 0**

**i . e A B = 0 , but A ≠ 0 , B ≠ 0**

**Ex :-**

$$B = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} , A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

$$AB= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

**( 3 ) IF A . B = A . C that is not necessary to be B = C**

**Ex :-**

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} , B = \begin{pmatrix} 4 & 2 \\ 6 & 4 \end{pmatrix} , C = \begin{pmatrix} -4 & 14 \\ 10 & -2 \end{pmatrix}$$

$$A\,B = \begin{pmatrix} 32 & 20 \\ 64 & 40 \end{pmatrix} , \quad A\,C = \begin{pmatrix} 32 & 20 \\ 64 & 40 \end{pmatrix}$$

## Kinds of Matrices انواع المصفوفات

**(1)** ***zero matrix*** :- is a matrix all of whose elements are zero and is denoted by $O_{m*n}$

**Ex:-**

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O_{2*3}$$

$$O_{3*3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**(2)** ***square matrix*** :- is a matrix has the same number of rows and columns

**Ex:-**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

**Remarks :-**

(1) square matrix has ( ***main diagonal*** ) with its elements

$a_{11}$ , $a_{22}$ , $a_{33}$

(2) the ***trace*** of matrix is the sum of elements of the main diagonal

i.e if $A = (a_{ij})_{n*n}$ $T(A) = \sum a_{ii}$

$= a_{11} + a_{22} + \ldots\ldots\ldots + a_{nn}$

**Ex:-** let

$$A = \begin{pmatrix} -7 & 3 & 9 \\ 0 & 5 & -8 \\ 2 & 6 & 12 \end{pmatrix}$$ find T ( A )

sol :-

**T ( A )** = ( -7 ) + ( 5 ) + ( 12 ) = 10

**(3)** ***Diagonal matrix*** :- is a square matrix which all elements are zero except elements of diagonal

**Ex:-**

$$A = \begin{pmatrix} -7 & 0 & 0 \\ 0 & 33 & 0 \\ 0 & 0 & 44 \end{pmatrix}_{3*3}$$

**(4)** ***Row matrix*** :- a matrix with only one row and ( n ) columns

**Ex:-**

$$A = \begin{pmatrix} -3 & 0 & 7 & 0 & 1/9 & 2 & 4 \end{pmatrix}_{1*7}$$

**(5)** ***Column matrix*** :- a matrix with only one column and ( m) rows

**Ex:-**

$$A = \begin{pmatrix} 5 \\ 3 \\ 7 \\ 9 \\ 0 \\ 2 \end{pmatrix}_{6*1}$$

**(6)** ***Lower triangular matrix*** :- a square matrix A= ( a ij ) n*n such that

**aij = 0 for all i < j**

**Ex:-**

$$A = \begin{pmatrix} 22 & 0 & 0 \\ 7 & 8 & 0 \\ 0 & 9 & 2 \end{pmatrix}_{3*3}$$

**(7)** ***upper triangular matrix*** :- a square matrix A = ( a ij ) n * n such that

**aij = 0 for all i > j**

Ex:-

$$A = \begin{pmatrix} 33 & 2 & 7 \\ 0 & 4 & -55 \\ 0 & 0 & 1/9 \end{pmatrix}_{3*3}$$

**(8)** ***Identity matrix*** :- a square matrix of order n which every diagonal elements are equal to 1 and other elements are equal to 0 denoted by In

**Ex:-**

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3*3}$$

***Note*** :- let A n*n be a matrix and In be identity matrix then

**A .I = I .A = A**

## Special Matrices المصفوفات الخاصة

### (1) Periodic Matrix المصفوفة الدورية

**Def** :- A matrix A such that $A^{K+1} = A$ is called ***periodic matrix***, where **K** is positive integer number .

Ex :- Show that A is periodic matrix of degree two

$$A = \begin{pmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & 3 \end{pmatrix}_{3*3}$$

### (2)Idempotent Matrix مصفوفه متساوية القوى

**Def** :- A matrix A such that $A^2 = A$ is called ***Idempotent matrix***

**Ex :-** Zero matrix is idempotent matrix .

***H.W*** :- Give an example of an idempotent and periodic matrix .

### (3) Nilpotent Matrix المصفوفة معدومة القوى

**Def** :- A matrix A such that $A^P = 0$ is called ***Nilpotent matrix***, where P is positive integer number .

Ex :- Show that A is nilpotent matrix of degree two

$$A=\begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix}_{3*3}$$

Sol:-

$$A^2=\begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix} . \begin{pmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$A^2 = A.A = 0$

## (4)Transpo se Of Matrix منقول المصفوفة ( مدور المصفوفة )

**Def** :- If A=(aij)n*m then the matrix $A^T$ **=(aji)m*n** obtained by interchanging rows and columns is called ***Transpose*** of A

**ملاحظة :-** اي ان $A^T$ هي مصفوفة ناتجة من استبدال الاعمدة بدل الصفوف وبالعكس في المصفوفة A .

Ex :-

$$A=\begin{pmatrix} 6 & 1 & -14 & 8 \\ -3 & 0 & 5 & 7 \\ 0 & -1/4 & 2 & 0 \end{pmatrix}_{3*4}$$

$$A^T=\begin{pmatrix} 6 & -3 & 0 \\ 1 & 0 & -1/4 \\ -14 & 5 & 2 \\ 8 & 7 & 0 \end{pmatrix}_{4*3}$$

**Theorem :- if $A^T$ and $B^T$ are transpose of A and B , and if K is any scalar number then :-**

**(1) $(A^T)^T = A$**

**(2) $(A+B)^T = A^T + B^T$**

**(3) $(AB)^T = B^T . A^T$**

**(4) $(kA)^T = k A^T$**

**Proof** **البرهان :-**

(1) let $A=(a_{ij})_{m*n}$

**L . H . S** ناخذ الطرف الايسر

$(A^T)^T = ((a_{ij})^T_{m*n})^T$

$= ((a_{ji})_{n*m})^T$ by $\{ (a_{ij})^T_{m*n} = (a_{ji})_{n*m} \}$

$= (a_{ij})_{m*n}$ by $\{ (a_{ij})^T_{m*n} = (a_{ji})_{n*m} \}$

$= A$

**(2) let $A=(a_{ij})_{m*n}$ , $B=(b_{ij})_{m*n}$**

then $A^T=(a_{ji})_{n*m}$ , $B^T=(b_{ji})_{n*m}$ by $\{ (a_{ij})^T_{m*n} = (a_{ji})_{n*m} \}$

**L . H . S** ناخذ الطرف الايسر

$(A+B)^T = [ (a_{ij})_{m*n} + (b_{ij})_{m*n} ]^T$

$= [ (a_{ij} + b_{ij})_{m*n} ]^T$ by $\{ (a_{ij})_{m*n} + (b_{ij})_{m*n} = (a_{ij} + b_{ij})_{m*n}$

$= [ (c_{ij})_{m*n} ]^T$

by $\{ (a_{ij})^T_{m*n} = (a_{ji})_{n*m} \}$

$= [ (c_{ji})_{n*m} ]$

$= [ (a_{ji} + b_{ji})_{n*m} ]$

$= (a_{ji})_{n*m} + (b_{ji})_{n*m}$

$= A^T + B^T$

.***H .W*** :- prove ( 3,4)

## (5)Symmetric Matrix المصفوفة المتنا ظرة

**Def** :- A square matrix A such that $A = A^T$ is called ***Symmetric Matrix.***

**Ex:- Define and given an example of a symmetric matrix**

**Sol :-**

$$A=\begin{pmatrix} 3 & 7 & 2 \\ 7 & 8 & 3 \\ 2 & 3 & 9 \end{pmatrix}, \quad A^T = \begin{pmatrix} 3 & 7 & 2 \\ 7 & 8 & 3 \\ 2 & 3 & 9 \end{pmatrix}$$

$A = A^T$

Then A Symmetric matrix .

$$B = \begin{pmatrix} -6 & 3 & -5 \\ 3 & 8 & -9 \\ 5 & -9 & 0 \end{pmatrix}, \quad B^T = \begin{pmatrix} -6 & 3 & 5 \\ 3 & 8 & -9 \\ -5 & -9 & 0 \end{pmatrix}$$

$B \neq B^T$

Then B is not Symmetric matrix .

**Theorem :- if A square matrix , prove that $A+A^T$ is symmetric matrix .**

**Proof :-**

We must prove that

$[A+A^T]^T = [A + A^T]$

L . H . S

$[A + A^T]^T = [A^T + (A^T)^T]$ , by $(A+B)^T = A^T + B^T$

$= A^T + A$ , by $(A^T)^T = A$

$= A + A^T$ , by $A + B = B + A$

Then $A + A^T$ is Symmetric matrix

## (6)Skew -Symmetric Matrix المصفوفة المتناظرة عكسياً

**Def** :- A square matrix A such that $A = -A^T$ is called *Skew-Symmetric Matrix.*

**Ex :-**

$$A = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix}, \quad A^T = \begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix}, \quad -A^T = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix}$$

$A = -A^T$

$$B = \begin{pmatrix} 0 & 1/3 & 5 \\ -1/3 & 0 & 7 \\ -5 & -7 & 0 \end{pmatrix}, \quad B^T = \begin{pmatrix} 0 & -1/3 & -5 \\ 1/3 & 0 & -7 \\ 5 & 7 & 0 \end{pmatrix}$$

$$-B^T = \begin{pmatrix} 0 & 1/3 & 5 \\ -1/3 & 0 & 7 \\ -5 & -7 & 0 \end{pmatrix}$$

$\mathbf{B = -B^T}$

Then **B Skew-Symmetric** matrix .

**ملاحظة :-** عند وضع مثال عن مصفوفة متناظرة عكسياً يجب ان تكون عناصر القطر الرئيسي تساوي صفر لكي لاتتغير الاشارة .

**H . W** :- (1) Give an example of Symmetric and Skew-Symmetric matrix such that the order of the matrix is (3*3) and (4*4) .

**(2)*Theorem* :- if A square matrix , prove that $A-A^T$ is skew-symmetric matrix .**

## (7)Orthogonal Matrix المصفوفة المتعامدة

**Def** :- A square matrix A such that $A.A^T = A^T.A = I$ is called ***Orthogonal Matrix*** .

**Ex :-**

$$A=\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}_{3*3}$$

**Sol :-**

$$A^T=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}_{3*3}$$

$$A.A^T=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3*3}$$

$$A^T.A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3*3}$$

$$\mathbf{A.A^T = A^T.A = I}$$

**H . W** :- (1) Give an example of ***Orthogonal Matrix*** such that the . order of the matrix is (3*3) and (4*4) .

(2) Give an example of **Orthogonal matrix** of the order (3*3) and (4*4).

(3) Solve Exc- ( 1.3 ) Page ( 33) ?