Example 10.4: Evaluate $\int_{-1}^{1}\int_{0}^{y} xye^{x^2}\,dx\,dy$.

Solution:

$$\int_{-1}^{1}\int_{0}^{y} xye^{x^2}\,dx\,dy = \int_{-1}^{1}\frac{1}{2}\int_{0}^{y} 2xye^{x^2}\,dx\,dy$$

$$= \int_{-1}^{1}\frac{1}{2}\left[ye^{x^2}\right]_{x=0}^{x=y} dy$$

$$= \int_{-1}^{1}\left(\frac{y}{2}e^{y^2} - \frac{y}{2}\right) dy$$

$$= \int_{-1}^{1}\frac{2y}{4}e^{y^2}\,dy - \int_{-1}^{1}\frac{y}{2}\,dy$$

$$= \left[\frac{e^{y^2}}{4}\right]_{-1}^{1} - \left[\frac{y^2}{4}\right]_{-1}^{1}$$

$$= \left(\frac{e}{4} - \frac{e}{4}\right) - \left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

Double integrals as volumes 10.5:

If $f(x,y) \geq 0$ over a region R, then

$$\iint_R f(x,y)\,dA = \text{The volume of the solid}$$

which is bounded below by the region R and above by the surface $z = f(x,y)$.

Example 10.6: Find the volume under the plane $z = 4 - x - y$ over the rectangular region $R: 0 \leq x \leq 2, 0 \leq y \leq 1$ in the xy-plane.

Solution:

$$\text{Volume} = \iint_R f(x,y)\, dA = \iint_R z\, dy\, dx$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\, dy\, dx$$

$$= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left(\left(4 - x - \frac{1}{2}\right) - 0 \right) dx$$

$$= \int_0^2 \left(\frac{7}{2} - x \right) dx$$

$$= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2$$

$$= \left(\frac{7}{2}(2) - \frac{4}{2} \right) - 0 = 7 - 2 = 5.$$

Fubini's Theorem (First Form) 10.7:

If $f(x,y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx .$$

Example 10.8: Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 1-6x^2y$ and $R: 0 \le x \le 2$, $-1 \le y \le 1$.

Solution: By Fubini's theorem

$$\iint_R f(x,y)\,dA = \int_{-1}^{1} \int_0^2 (1-6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1} \left[x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} ((2-16y)-0)\,dy$$

$$= \left[2y - 8y^2\right]_{-1}^{1} = (2-8)-(-2-8) = -6+10 = 4.$$

Reversing the order of integration gives the same answer:

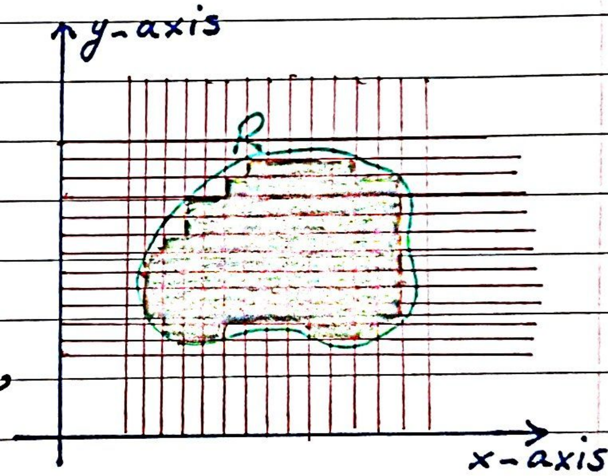
$$\int_0^2 \int_{-1}^{1} (1-6x^2y)\,dy\,dx = \int_0^2 \left[y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 ((1-3x^2) - (-1-3x^2))\, dx = \int_0^2 2\, dx = \Big[2x\Big]_0^2$$

$= 4 - 0 = 4.$

Double Integrals over a Bounded Nonrectangular Region 10.9:

Let $f(x,y)$ be a function defined on a bounded nonrectangular region R, and let R be divided by straight lines parallel to the x-axis and straight lines parallel to the y-axis such that these lines cover R by rectangular pieces of area $\Delta A = \Delta x \cdot \Delta y$.



Then number the rectangular pieces of area $\Delta A = \Delta x \cdot \Delta y$ that lies entirely within the region (shaded in the figure) in some order $\Delta A_1, \Delta A_2, \dots, \Delta A_n$, and choose a point (x_k, y_k) in each rectangular piece of area ΔA_k, and form the following sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \cdot \Delta A_k \quad .$$

If f is continuous on R, and as we reduce the values of Δx and Δy and let them approach to zero, then the sums S_n will approach a limit called the double integral of f

over R. The notation for it is

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy \text{ , i.e}$$

$$\iint_R f(x,y)\,dA = \lim_{\Delta A_k \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k .$$

Theorem 10.10 (Fubini's Theorem (Stronger Form)):

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx .$$

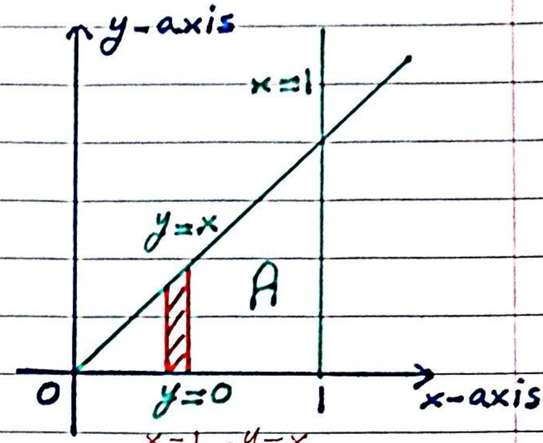
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy .$$

Example 10.11: Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$, and whose top lies in the plane $z = f(x,y) = 3 - x - y$.

Solution:

1. $$V=\int_0^1\int_{y=0}^{y=x}(3-x-y)\,dy\,dx$$



$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x,y)\,dy\,dx$$

$$=\int_0^1\left[3y-xy-\frac{y^2}{2}\right]_{y=0}^{y=x}dx$$

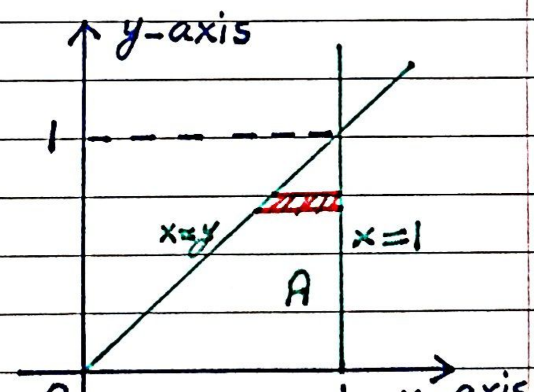
$$=\int_0^1\left(3x-x^2-\frac{x^2}{2}\right)dx$$

$$=\int_0^1\left(3x-\frac{3}{2}x^2\right)dx$$

$$=\frac{3x^2}{2}-\frac{x^3}{2}\Big]_0^1=\frac{3}{2}-\frac{1}{2}=1.$$

2. When the order of the integration is reversed, the integral for the volume is

$$V=\int_0^1\int_{x=y}^{x=1}(3-x-y)\,dx\,dy$$



$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} f(x,y)\,dx\,dy$$

$$=\int_0^1\left[3x-\frac{x^2}{2}-xy\right]_{x=y}^{x=1}dy$$

$$=\int_0^1\left(\left(3-\frac{1}{2}-y\right)-\left(3y-\frac{y^2}{2}-y^2\right)\right)dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy$$

$$= \frac{5}{2}y - 2y^2 + \frac{1}{2}y^3 \Big]_0^1 = \frac{5}{2} - 2 + \frac{1}{2} = 1 .$$

Double integrals as areas 10.12 :

If we take $f(x,y) = 1$ in theorem 10.10, then

1. the area of the region R defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a,b]$ is

$$\text{The area of } R = \iint_R dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx ,$$

2. the area of the region R defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c,d]$ is

$$\text{The area of } R = \iint_R dA = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy .$$

Example 10.13 : Find the area of the region R bounded by $y = x$ and $y = x^2$, in the first quadrant.

Solution :

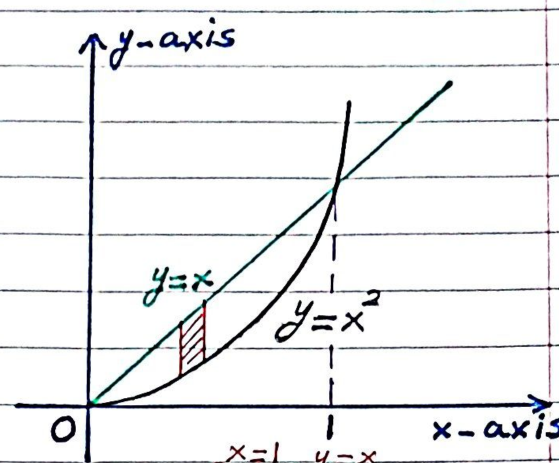
The area A of the region R is

$$A=\int_0^1\int_{y=x^2}^{y=x} dy\,dx$$

$$=\int_0^1 [y]_{y=x^2}^{y=x}\,dx$$

$$=\int_0^1 (x-x^2)\,dx$$

$$=\frac{x^2}{2}-\frac{x^3}{3}\Big]_0^1=\frac{1}{2}-\frac{1}{3}=\frac{3-2}{6}=\frac{1}{6}.$$



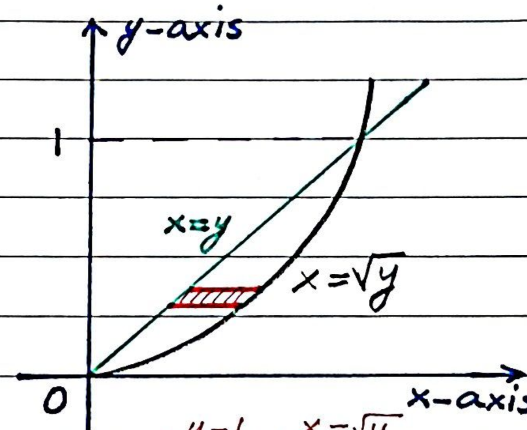
$$A=\int_{x=0}^{x=1}\int_{y=x^2}^{y=x} dy\,dx$$

We can find the area also as follows :

$$A=\int_0^1\int_{x=y}^{x=\sqrt{y}} dx\,dy$$

$$=\int_0^1 [x]_{x=y}^{x=\sqrt{y}}\,dy$$

$$=\int_0^1 (\sqrt{y}-y)\,dy$$



$$A=\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} dx\,dy$$

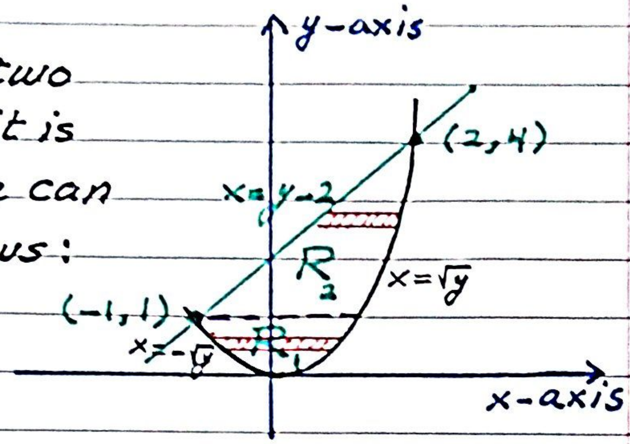
$$=\frac{y^{\frac{3}{2}}}{\frac{3}{2}}-\frac{y^2}{2}\Big]_0^1=\frac{1}{\frac{3}{2}}-\frac{1}{2}=\frac{2}{3}-\frac{1}{2}=\frac{4-3}{6}=\frac{1}{6}.$$

Example 10.14 :

Find the area of the region enclosed by the parabola $y=x^2$ and the line $y=x+2$.

Solution:

If we divide R into two regions R_1 and R_2 as it is shown in the figure we can find the area as follows:



$$A=\iint_{R_1} dA+\iint_{R_2} dA$$

$$=\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy+\int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy$$

$$=\int_0^1 \left[x\right]_{-\sqrt{y}}^{\sqrt{y}} dy+\int_1^4 \left[x\right]_{y-2}^{\sqrt{y}} dy$$

$$=\int_0^1 (\sqrt{y}+\sqrt{y})\,dy+\int_1^4 (\sqrt{y}-y+2)\,dy$$

$$=\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}+\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1+\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}-\frac{y^2}{2}+2y\right]_1^4$$

$$=\left[\frac{4}{3}y^{\frac{3}{2}}\right]_0^1+\left[\frac{2}{3}y^{\frac{3}{2}}-\frac{y^2}{2}+2y\right]_1^4$$

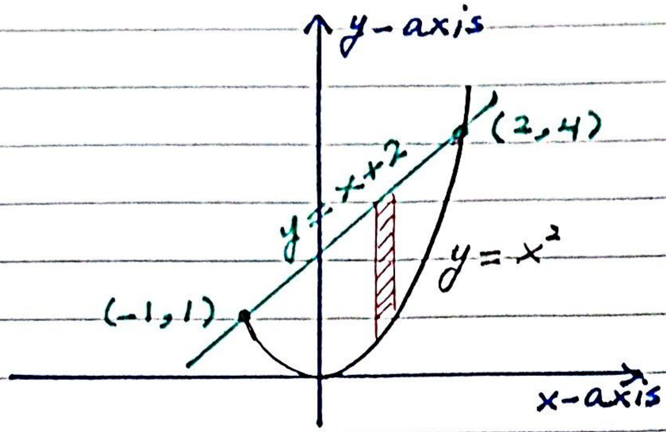
$$=\frac{4}{3}-0+\left(\frac{16}{3}-\frac{16}{2}+8\right)-\left(\frac{2}{3}-\frac{1}{2}+2\right)$$

$$=\frac{4}{3}+\frac{16}{3}-8+8-\frac{2}{3}+\frac{1}{2}-2=\frac{18}{3}-\frac{3}{2}$$

$$=\frac{9}{2}.$$

We can find the area also as follows:

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx$$



$$= \int_{-1}^{2} [y]_{x^2}^{x+2}\, dx$$

$$= \int_{-1}^{2} (x+2-x^2)\, dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big]_{-1}^{2}$$

$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 6 - \frac{8}{3} + \frac{3}{2} - \frac{1}{3} = \frac{15}{2} - 3 = \frac{9}{2}.$$