§8 : Lagrange Multipliers

The method of Lagrange Multipliers 8.1:
Suppose that $f(x,y,z)$ and $g(x,y,z)$ have continuous partial derivatives. The local maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$ are at the points (x,y,z) that satisfy the equations

$$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \text{and} \quad g(x,y,z)=0 \quad \text{simultaneously.}$$

(i.e. find the values of x, y, z, and λ that satisfy the equations
$\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x,y,z)=0$ simultaneously).

Definition 8.2: The scalar λ in the vector equation $\vec{\nabla} f = \lambda \vec{\nabla} g$ is called Lagrange multiplier.

Remark 8.3: $\vec{\nabla} f = \lambda \vec{\nabla} g$ implies that

$$f_x = \lambda g_x \;, \quad f_y = \lambda g_y \;, \quad \text{and} \quad f_z = \lambda g_z .$$

Example 8.4: Find the minimum value of the function $f(x,y,z) = 3x+2y+z+5$ subject to the constraint $g(x,y,z) = 9x^2+4y^2-z=0$.

Solution:

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = 3\vec{i} + 2\vec{j} + \vec{k} \quad \text{and}$$

$\vec{\nabla} g = g_x \vec{i} + g_y \vec{j} + g_z \vec{k} = 18x \vec{i} + 8y \vec{j} - \vec{k}$

$\therefore \vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 3\vec{i} + 2\vec{j} + \vec{k} = \lambda(18x\vec{i} + 8y\vec{j} - \vec{k})$

$\Rightarrow 3 = \lambda(18x) \quad , 2 = \lambda(8y) , \ 1 = -\lambda$

$\Rightarrow 3 = -18x \quad , 2 = -8y \Rightarrow x = \frac{3}{-18} = -\frac{1}{6}$ and

$y = \frac{2}{-8} = -\frac{1}{4}$.

$\therefore g(x,y,z) = 9x^2 + 4y^2 - z = 9\left(\frac{1}{36}\right) + 4\left(\frac{1}{16}\right) - z = 0$

$\Rightarrow z = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\therefore f\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right) = 3\left(-\frac{1}{6}\right) + 2\left(-\frac{1}{4}\right) + \frac{1}{2} + 5 = \frac{9}{2}$

is the minimum value of the function $f(x,y,z)$ at the point $\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right)$.

Example 8.5: Use the method of Lagrange multipliers to find:

i- the minimum value of $x+y$ subject to the constraint $xy = 16$, $x > 0$, $y > 0$.

ii- the maximum value of xy subject to the constraint $x+y = 16$.

Solution:

i- $f(x,y) = x+y \quad , g(x,y) = xy - 16 = 0 \Rightarrow$

$\vec{\nabla} f = \vec{i} + \vec{j} \quad , \vec{\nabla} g = y\vec{i} + x\vec{j}$.

$\therefore \vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \vec{i} + \vec{j} = \lambda(y\vec{i} + x\vec{j})$

$\Rightarrow \lambda y = 1$ and $\lambda x = 1$

$\Rightarrow \lambda = \frac{1}{y}$ and $\lambda x = 1$

$\Rightarrow \frac{1}{y}(x) = 1 \Rightarrow \frac{x}{y} = 1 \Rightarrow x = y$

$\therefore g(x,y) = x^2 - 16 = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

and $y = x \Rightarrow$ the points are $(4,4)$ and $(-4,-4)$.

$\therefore f$ has minimum value at the point $(4,4)$ and the minimum value is $f(4,4) = 4+4 = 8$.

ii- $f(x,y) = xy$, $g(x,y) = x+y-16 = 0 \Rightarrow$

$\vec{\nabla} f = y\vec{i} + x\vec{j}$, $\vec{\nabla} g = \vec{i} + \vec{j}$.

$\therefore \vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow y\vec{i} + x\vec{j} = \lambda(\vec{i} + \vec{j})$

$\Rightarrow y = \lambda$ and $x = \lambda \Rightarrow x = y$.

$\therefore g(x,y) = x + x - 16 = 0 \Rightarrow 2x = 16 \Rightarrow x = 8$ and $y = 8 \Rightarrow f$ has maximum value at the point $(8,8)$.

$\therefore f(8,8) = 8(8) = 64$ is the maximum value of f at the point $(8,8)$.

§9: Implicit Differentiation

Theorem 9.1: Suppose that the equation $F(x,y,z)=0$ determines z as a differentiable function of x and y. Then at points where $F_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example 9.2: Suppose that x, y, and z are variables and z is a function of x and y satisfying that $2x^2+y^2+z^2-26=0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(2,-3,3)$.

Solution: $F(x,y,z) = 2x^2+y^2+z^2-26=0 \Rightarrow$

$F_x = 4x$, $F_y = 2y$, and $F_z = 2z$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4x}{2z} = -2\frac{x}{z} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y}{2z} = -\frac{y}{z}$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(2,-3,3)} = -2\left(\frac{2}{3}\right) = -\frac{4}{3} \quad \text{and}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,-3,3)} = -\left(\frac{-3}{3}\right) = 1.$$

Example 9.3: Suppose that x, y, and z are variables and z is a function of x and y satisfying that $x^3+y^3+z^3+3xyz=2$.

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(2,1,-1)$.

Solution: $F(x,y,z)=x^3+y^3+z^3+3xyz-2=0$

$$\Rightarrow F_x=3x^2+3yz\ ,\ F_y=3y^2+3xz\ ,\ F_z=3z^2+3xy$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+3yz}{3z^2+3xy}=-\frac{x^2+yz}{z^2+xy}\ ,\text{ and}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{3y^2+3xz}{3z^2+3xy}=-\frac{y^2+xz}{z^2+xy}$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(2,1,-1)}=-\frac{2^2+1(-1)}{(-1)^2+2(1)}=-\frac{3}{3}=-1 \text{ and}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,1,-1)}=-\frac{1^2+2(-1)}{(-1)^2+2(1)}=-\frac{-1}{3}=\frac{1}{3}$$

Example 9.4: Suppose that r, s, t, and w are variables and w is a function of r, s, and t satisfying that $e^{rt}-2se^{w}+wt-3w^2r=5$. Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$, and $\frac{\partial w}{\partial t}$.

Solution: $F(r,s,t,w)=e^{rt}-2se^{w}+wt-3w^2r-5=0$

$$\Rightarrow F_r=te^{rt}-3w^2\ ,\ F_s=-2e^{w}\ ,\ F_t=re^{rt}+w\ ,$$

and $F_\omega = -2se^\omega + t - 6\omega r$

$$\Rightarrow \frac{\partial \omega}{\partial r} = -\frac{F_r}{F_\omega} = \frac{-te^{rt} + 3\omega^2}{-2se^\omega + t - 6\omega r},$$

$$\frac{\partial \omega}{\partial s} = -\frac{F_s}{F_\omega} = \frac{2e^\omega}{-2se^\omega + t - 6\omega r}, \text{ and}$$

$$\frac{\partial \omega}{\partial t} = -\frac{F_t}{F_\omega} = \frac{re^{rt} + \omega}{2se^\omega - t + 6\omega r}.$$

§ 10 : Double Integrals

Double Integrals over Rectangles 10.1:

Let $f(x,y)$ be a function defined on a rectangular region R given by

$$R: \ a \leq x \leq b \ , \ c \leq y \leq d$$

and let R be divided by straight lines parallel to the x-axis and straight lines parallel to the y-axis and these lines divide R into small rectangular pieces of area $\Delta A = \Delta x . \Delta y$. Then number these rectangular pieces in some order $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, and choose a point (x_k, y_k) in each rectangular piece of area ΔA_k , and form the following sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) . \Delta A_k \ .$$

If f is continuous on R, and as we reduce the values of Δx and Δy and let them approach to zero, then the sums S_n will approach a limit called the double integral of f over R. The notation for it is

$$\iint_R f(x,y) \, dA \quad \text{or} \quad \iint_R f(x,y) \, dx \, dy \ , \text{i.e.}$$

$$\iint_R f(x,y)\,dA = \lim_{\Delta A_k \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k .$$

Properties of the Double Integrals 10.2:

Double integrals of continuous functions over a region R have the following algebraic properties that are useful in calculations and applications.

1) $\iint_R k f(x,y)\,dA = k \iint_R f(x,y)\,dA$, (k is any number)

2) $\iint_R (f(x,y) + g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA .$

3) $\iint_R (f(x,y) - g(x,y))\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA .$

4) $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

5) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y)$

$\geq g(x,y)$ on R.

6) $$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

where R is the union of two nonoverlapping regions R_1 and R_2.

Example 10.3: Evaluate $\int_0^1 \int_x^{x-1} (x^2+e^y)\,dy\,dx$.

Solution:

$$\int_0^1 \int_x^{x-1} (x^2+e^y)\,dy\,dx = \int_0^1 \Big[x^2y+e^y\Big]_{y=x}^{y=x-1} dx$$

$$= \int_0^1 \Big(\big(x^2(x-1)+e^{x-1}\big)-\big(x^3+e^x\big)\Big)\,dx$$

$$= \int_0^1 (\cancel{x^3}-x^2+e^{x-1}-\cancel{x^3}-e^x)\,dx$$

$$= \int_0^1 (-x^2+e^{x-1}-e^x)\,dx$$

$$= \Big[-\frac{x^3}{3}+e^{x-1}-e^x\Big]_0^1$$

$$= \Big(-\frac{1}{3}+e^0-e\Big)-\big(0+e^{-1}-e^0\big)$$

$$= -\frac{1}{3}+1-e-e^{-1}+1 = \frac{5}{3}-e-\frac{1}{e}.$$