Proposition 5.5: If $P_0(x_0, y_0, z_0)$ is a point on the surface $z=f(x,y)$, f_x and f_y are continuous at (x_0, y_0), then the tangent plane to the surface $z=f(x,y)$ at $P_0(x_0, y_0, f(x_0,y_0))$ is the plane whose equation is

$$f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)-(z-z_0)=0,$$

and the normal line to the surface $z=f(x,y)$ at $P_0(x_0, y_0, f(x_0,y_0))$ is the line whose equations are

$$x = x_0 + f_x(x_0,y_0)\,t \quad ,$$

$$y = y_0 + f_y(x_0,y_0)\,t \quad ,$$

$$z = z_0 - t \; .$$

If none of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ is zero, then the normal line is also given by the equations

$$\frac{x-x_0}{f_x(x_0,y_0)} = \frac{y-y_0}{f_y(x_0,y_0)} = \frac{z-z_0}{-1}$$

Example 5.6: Find the equation of the tangent plane and the equations of the normal line to the surface $z=f(x,y) = 8-x^2-y^2$ at the point $P_0(1,1,6)$.

Solution :

$$f_x(1,1) = f_x \big|_{(1,1)} = -2x \big|_{(1,1)} = -2 \ ,$$

$$f_y(1,1) = f_y \big|_{(1,1)} = -2y \big|_{(1,1)} = -2 \ .$$

$\therefore$ The equation of the tangent plane to the surface $z = f(x,y) = 8 - x^2 - y^2$ at the point $P_0(1,1,6)$ is

$-2(x-1) - 2(y-1) - (z-6) = 0$

$\Rightarrow -2x - 2y - z = -10$

$\Rightarrow 2x + 2y + z = 10$.

The equations of the normal line to the surface $z = f(x,y) = 8 - x^2 - y^2$ at the point $P_0(1,1,6)$ are

$x = 1 - 2t \ , \ y = 1 - 2t \ , \ z = 6 - t$ or

$$\frac{x-1}{-2} = \frac{y-1}{-2} = \frac{z-6}{-1} \ .$$

Remark 5.7: In Example 5.6 , we can rewrite $f(x,y) = z = 8 - x^2 - y^2$ in the form $h(x,y,z) = -x^2 - y^2 - z = -8$, which show that the surface $f(x,y) = z$ is the same as the level surface $h(x,y,z) = c$ (where $c = -8$).

Another Solution of Example 5.6:

$$h_x(1,1,6) = -2x \big|_{(1,1,6)} = -2 \ ,$$

$$h_y(1,1,6) = -2y \big|_{(1,1,6)} = -2 \ ,$$

$$h_z(1,1,6) = -1 \big|_{(1,1,6)} = -1 \ .$$

$\therefore$ The equation of the tangent plane to the surface $f(x,y)=z$ (which is the same as the level surface $h(x,y,z)=-8$) is
$h_x(1,1,6)\cdot(x-1)+h_y(1,1,6)\cdot(y-1)+h_z(1,1,6)\cdot(z-6)$
$=0$

$\Rightarrow -2(x-1)-2(y-1)-1(z-6)=0$

$\Rightarrow 2x+2y+z=10$.

The equations of the normal line to the surface $f(x,y)=z$ (which is the same as the level surface $h(x,y,z)=-8$) are

$$\frac{x-1}{-2}=\frac{y-1}{-2}=\frac{z-6}{-1} \quad \text{or}$$

$$x=1-2t,\ y=1-2t,\ z=6-t.$$

§6: Higher Order Derivatives

Remarks 6.1:

1) The second order partial derivatives, which are denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xx}, f_{yy}, f_{yx}, f_{xy} respectively are defined by the equations:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = f_{xx},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = f_{yx},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = f_{xy}.$$

Note that the order in which the derivatives are taken to evaluate $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$ is to differentiate f first with respect to y, then differentiate the result with respect to x.

2) If $f(x,y)$ have continuous partial derivatives, then $f_{xy} = f_{yx}$.

3) The third order partial derivatives, which are denoted by $\frac{\partial^3 f}{\partial x^3}$, $\frac{\partial^3 f}{\partial y^3}$,

$\frac{\partial^3 f}{\partial x^2 \partial y}$, $\frac{\partial^3 f}{\partial y \partial x^2}$, $\frac{\partial^3 f}{\partial x \partial y^2}$, $\frac{\partial^3 f}{\partial y^2 \partial x}$,

$\frac{\partial^3 f}{\partial x \partial y \partial x}$, $\frac{\partial^3 f}{\partial y \partial x \partial y}$ or f_{xxx}, f_{yyy}, f_{yxx},

f_{xxy}, f_{yyx}, f_{xyy}, f_{xyx}, f_{yxy} respectively are defined by

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(f_x)\right) = \frac{\partial}{\partial x}(f_{xx})$$
$$= f_{xxx},$$

$$\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(f_y)\right) = \frac{\partial}{\partial y}(f_{yy})$$
$$= f_{yyy},$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(f_y)\right) = \frac{\partial}{\partial x}(f_{yx})$$
$$= f_{yxx},$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(f_x)\right) = \frac{\partial}{\partial y}(f_{xx})$$
$$= f_{xxy}, \ldots.$$

4) If $f(x,y)$ have continuous partial derivatives, then $f_{xxy} = f_{xyx} = f_{yxx}$ and $f_{yyx} = f_{yxy} = f_{xyy}$.

5) If all the partial derivatives of $f(x,y)$ are continuous, the notation

$\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ may be used to denote the result of differentiation of the function $f(x,y)$, n-times with respect to y and m-times with respect to x.

6) If $f(x,y,z,s,u,v)$ have continuous partial derivatives, then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{xu} = f_{ux}$, $f_{uz} = f_{zu}$, ...

7) $$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^4 f}{\partial x \partial y \partial x \partial y} = \frac{\partial^4 f}{\partial x \partial y^2 \partial x} =$$

$$\frac{\partial^4 f}{\partial y \partial x^2 \partial y} = \frac{\partial^4 f}{\partial y \partial x \partial y \partial x} = \frac{\partial^4 f}{\partial y^2 \partial x^2}.$$

Example 6.2: Given that $z = x^3 + 3x^2y - 2x^2y^2 - y^4 + 3xy$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^3 z}{\partial x \partial y^2}$, $\frac{\partial^4 z}{\partial x^2 \partial y^2}$.

Solution:

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy - 4xy^2 + 3y.$$

$$\frac{\partial z}{\partial y} = 3x^2 - 4x^2y - 4y^3 + 3x .$$

$$\frac{\partial^2 z}{\partial x^2} = 6x + 6y - 4y^2 .$$

$$\frac{\partial^2 z}{\partial y^2} = -4x^2 - 12y^2 .$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x - 8xy + 3 .$$

$$\frac{\partial^2 z}{\partial y \partial x} = 6x - 8xy + 3 .$$

$$\frac{\partial^3 z}{\partial x \partial y^2} = -8x .$$

$$\frac{\partial^4 z}{\partial x^2 \partial y^2} = -8 .$$

Exercises 6.3 :

1) Let $f(x,y) = x\cos y + ye^x$.

a- Find all the first partial derivatives of f.

b- Find all the second partial derivatives of f.

c- Find $\frac{\partial^3 f}{\partial x^2 \partial y}$ and $\frac{\partial^3 f}{\partial x \partial y^2}$.

2) Given that $u = e^x \cos y + e^x \sin z$.

Verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x} .$$

§ 7: Maximum, Minimum and Saddle Points

Definition 7.1: Let $z = f(x,y)$ be a function defined on a region R containing a point (a,b). Then $f(a,b)$ is said to be a relative maximum (or a local maximum) value of f if $f(a,b) \geq f(x,y)$ for all points $(x,y) \in R$ inside some circle whose center is (a,b).

Definition 7.2: Let $z = f(x,y)$ be a function defined on a region R containing a point (a,b). Then $f(a,b)$ is said to be a relative minimum (or a local minimum) value of f if $f(a,b) \leq f(x,y)$ for all points $(x,y) \in R$ inside some circle whose center is (a,b).

Definition 7.3: A point $(x, y, f(x,y))$ in the space R^3 is said to be a saddle point if it looks like a maximum point in some plane containing it and it looks like a minimum point in another plane containing it.

Definition 7.4: An interior point (a,b) of the region R (where R is the domain of the function $f(x,y)$) where both $f_x(a,b)$ and $f_y(a,b)$ are zero or where one or both $f_x(a,b)$ and $f_y(a,b)$ do not exist is called a critical point.

Testing 7.5 (Testing for Extreme Values):

Let $z=f(x,y)$ be continuous function and has continuous first and second partial derivatives on some region H of the xy-plane. Then f has local maximum (relative maximum), local minimum (relative minimum), saddle point at

1) The boundary of H (if H has a boundary).
2) The interior points of H where $f_x = f_y = 0$, or points where f_x or f_y not exist (the critical points).

Theorem 7.6 (Second Derivative Test for Local Extreme Values):

Suppose that $f(x,y)$ and its first and second partial derivatives are continuous inside some circle whose center is (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then

i) f has a local maximum (relative maximum) at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

ii) f has a local minimum (relative minimum) at (a,b) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

iii) f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b).

iv) The test gives no information if

$f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b).

Remark 7.7: The expression $f_{xx} f_{yy} - f_{xy}^2$ can be written as follows

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}, \quad \left(\text{i.e. } f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}\right).$$

Example 7.8: Find the local extreme values of the function $f(x,y) = 5x^2 + 4y^2 + 12$.

Solution:

The domain of f has no boundary points since f is defined on all the points of the xy-plane (which means that the domain is the xy-plane).

The derivatives $f_x = 10x$ and $f_y = 8y$ exist everywhere.

$f_x = 10x = 0$ and $f_y = 8y = 0 \Rightarrow x = y = 0$

$\therefore$ we have extreme value only at the point $(0,0)$.

Since $f_{xx} = 10$, $f_{yy} = 8$, $f_{xy} = 0$, then

$$f_{xx} f_{yy} - f_{xy}^2 = 10(8) - 0^2 = 80 > 0 \text{ and } f_{xx} > 0,$$

which implies that $f(0,0) = 12$ is a local minimum of f at $(0,0)$.

Example 7.9: Find the local extreme values of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Solution:

The function f is defined and differentiable for all x and y and its domain is the xy-plane which has no boundary points. Therefore f has extreme values only at the points where f_x and f_y are both zero.

$f_x = y - 2x - 2 = 0 \Rightarrow y - 2x - 2 = 0 \quad \ldots (1)$

$f_y = x - 2y - 2 = 0 \Rightarrow x - 2y - 2 = 0 \quad \ldots (2)$

(1) implies that $y = 2x + 2 \quad \ldots (3)$
substitute (3) in (2) we get that
$x - 2(2x+2) - 2 = 0 \Rightarrow x - 4x - 6 = 0$
$\Rightarrow -3x = 6 \Rightarrow x = -2 \quad \ldots (4)$
substitute (4) in (3) we get that
$y = 2(-2) + 2 = -4 + 2 = -2$
$\therefore$ the point $(-2,-2)$ is the only point where f has an extreme value.
Since $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 1$, then

$f_{xx} f_{yy} - f_{xy}^2 = -2(-2) - 1^2 = 4 - 1 = 3 > 0$ and

$f_{xx} < 0$, which implies that $f(-2,-2) = 8$ is a local maximum of f at $(-2,-2)$.

Example 7.10: Find the local extreme values of the function $f(x,y) = 8xy - x$.
Solution:
Since the function f is defined and differentiable for all x and y and its domain the xy-plane has no boundary points, then the function f has extreme

values only at the points where $f_x = f_y = 0$.

$f_x = 8y - 1 = 0 \Rightarrow 8y = 1 \Rightarrow y = \frac{1}{8}$

$f_y = 8x = 0 \Rightarrow x = 0$

$\therefore$ the point $(0, \frac{1}{8})$ is the only point where f has an extreme value.

Since $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 8$, then

$f_{xx} f_{yy} - f_{xy}^2 = 0(0) - 8^2 = -64 < 0$, which

implies that f has a saddle point at $(0, \frac{1}{8})$, and f has no local extreme values.

Exercises 7.11: Find the local maximum, local minimum, and saddle points of each of the following functions:

1) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$.

2) $f(x,y) = x^2 + xy + 3x + 2y + 5$.

3) $f(x,y) = -2x^2 - y^2 - 2xy + 2x + 2y + 3$.