Remark 4.9: If f is a function of three variables x, y, and z, and let $\vec{u} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$ be a unit vector with its initial point $P_0(x_0, y_0, z_0)$ and making an angle α with the x-axis, an angle β with the y-axis, an angle γ with the z-axis, then the directional derivative of f at $P_0(x_0, y_0, z_0)$ in the direction $\vec{u}$ can be written as

$$\left(D_{\vec{u}} f\right)_{P_0} = f_x(x_0, y_0, z_0)\cos\alpha + f_y(x_0, y_0, z_0)\cos\beta + f_z(x_0, y_0, z_0)\cos\gamma$$

إذا كانت الدالة f لثلاث متغيرات x و y و z وأن $\vec{u} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$ هو متجه وحدة بدايته النقطة $P_0(x_0, y_0, z_0)$ صانعاً زاوية مقدارها α مع الـ x-axis وزاوية مقدارها β مع الـ y-axis وزاوية مقدارها γ مع الـ z-axis فإن المشتقة الاتجاهية للدالة f عند النقطة $P_0(x_0, y_0, z_0)$ باتجاه $\vec{u}$ يمكن كتابتها بالصيغة التالية

$$\left(D_{\vec{u}} f\right)_{P_0} = f_x(x_0, y_0, z_0)\cos\alpha + f_y(x_0, y_0, z_0)\cos\beta + f_z(x_0, y_0, z_0)\cos\gamma$$

Example 4.10: Find the directional derivative of $f(x, y, z) = 2xy + 3yz$ at the point $P_0(2, 4, -1)$ in the direction $\vec{A} = 2\vec{i} + 6\vec{j} - 3\vec{k}$.

Solution:

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\vec{i}+6\vec{j}-3\vec{k}}{\sqrt{4+36+9}}$$

$$= \frac{2}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{3}{7}\vec{k}$$

$$= \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}.$$

$f_x = 2y \Rightarrow f_x(2,4,-1) = 2(4) = 8$,

$f_y = 2x+3z \Rightarrow f_y(2,4,-1) = 2(2)+3(-1)$
$= 4-3 = 1$,

$f_z = 3y \Rightarrow f_z(2,4,-1) = 3(4) = 12$.

Therefore $\left(D_{\vec{u}} f\right)_{P_0} = f_x(x_0,y_0,z_0)\cos\alpha$
$+ f_y(x_0,y_0,z_0)\cos\beta + f_z(x_0,y_0,z_0)\cos\gamma$

$$= 8\left(\frac{2}{7}\right) + 1\left(\frac{6}{7}\right) + 12\left(-\frac{3}{7}\right)$$

$$= \frac{16}{7} + \frac{6}{7} - \frac{36}{7} = -\frac{14}{7} = -2.$$

Example 4.11: Estimate how much $f(x,y,z) = xe^y + yz$ will change (i.e. find Δf), if the point $P(x,y,z)$ is moved from $P_0(2,0,0)$ straight toward $P_1(4,1,-2)$ a distance of $\Delta s = 0.1$ units.

Solution:

$$\overrightarrow{P_0P_1} = (4-2)\vec{i} + (1-0)\vec{j} + (-2-0)\vec{k}$$

$$= 2\vec{i} + \vec{j} - 2\vec{k}.$$

$$\vec{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{2\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{9}}$$

$$= \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$$

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

$$= e^y \vec{i} + (x e^y + z)\vec{j} + y\vec{k}$$

$$(\vec{\nabla} f)_{P_0} = f_x(x_0, y_0, z_0)\vec{i} + f_y(x_0, y_0, z_0)\vec{j}$$

$$+ f_z(x_0, y_0, z_0)\vec{k}$$

$$= e^0 \vec{i} + (2e^0 + 0)\vec{j} + 0.\vec{k} = \vec{i} + 2\vec{j}.$$

Thus $(D_{\vec{u}} f)_{P_0} = (\vec{\nabla} f)_{P_0} \cdot \vec{u}$

$$= (\vec{i} + 2\vec{j}) \cdot \left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Therefore $\Delta f \approx (D_{\vec{u}} f)_{P_0} \cdot \Delta s$

$$\approx \frac{4}{3} \cdot 0.1$$

$$\approx 0.1333333 .$$

Exercise 4.12: Find $(D_{\vec{u}} f)_{P_0}$ of the following functions at P_0 in the direction of the given vector:

1. $f(x,y) = e^{x}\sin(y\pi)$, where $P_0(1,0)$ and $\vec{A} = 4\vec{i} + 4\vec{j}$.

2. $f(x,y,z) = e^{3x+4y}\cdot\cos 5z$, where $P_0(0,0,\frac{\pi}{6})$ and $\vec{B} = \vec{i} + \vec{j} - \vec{k}$.

3. $f(x,y) = \dfrac{x-y^2}{x}$, where $P_0(1,1)$ and $\vec{C} = 12\vec{i} - \vec{j}$.

4. $f(x,y,z) = \ln(x^2+y^2+z^2)$, where $P_0(2,1,0)$, $\vec{D} = 2\vec{i} - \vec{j} - 2\vec{k}$.

5. $f(x,y,z) = e^{x}\cos yz$, where $P_0(0,0,0)$ and $\vec{E} = 2\vec{i} + \vec{j} - 2\vec{k}$.

Proposition 4.13 (Properties of the Directional Derivative):

The properties of the directional derivative

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = |\nabla f| \cos\theta$ are:

1. The directional derivative has its largest positive value when $\cos\theta = 1$, or when $\vec{u}$ is the direction of the gradiant. That is, f

increases most rapidly in its domain in the direction of $\vec{\nabla} f$. The derivative in this direction is

$$D_{\vec{u}} f = |\vec{\nabla} f| \cos 0 = |\vec{\nabla} f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\vec{\nabla} f$. The derivative in this direction is

$$D_{\vec{u}} f = |\vec{\nabla} f| \cos \pi = -|\vec{\nabla} f|.$$

3. Any direction $\vec{u}$ perpendicular to the gradient is a direction of zero change in f since $D_{\vec{u}} f = |\vec{\nabla} f| \cos \frac{\pi}{2} = |\vec{\nabla} f| \cdot 0 = 0$.

Example 4.14: a) Find the derivative of $f(x,y) = 100 - x^2 - y^2$ at the point $P_0(3,4)$ in the direction of the unit vector $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$.

b) In what direction in its domain (the xy-plane) is f increasing most rapidly at P_0? What is the derivative of f in this direction?

c) Identify the directions in which the derivative of f is zero.

Solution:

$$f_x(3,4) = -2x \Big|_{(3,4)} = -6$$

$$f_y(3,4) = -2y \Big|_{(3,4)} = -8$$

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} = -2x\vec{i} - 2y\vec{j} \Rightarrow (\vec{\nabla} f)_{(3,4)} = -6\vec{i} - 8\vec{j}.$$

$$(D_{\vec{u}} f)_{(3,4)} = (\vec{\nabla} f)_{(3,4)} \cdot \vec{u}$$

$$= (-6\vec{i} - 8\vec{j})(u_1\vec{i} + u_2\vec{j}) = -6u_1 - 8u_2 .$$

b) The function increases most rapidly in the direction of the gradiant.
Since $|(\vec{\nabla} f)_{(3,4)}| = |-6\vec{i} - 8\vec{j}| = \sqrt{(-6)^2 + (-8)^2}$

$= \sqrt{36+64} = \sqrt{100} = 10$. Then the direction of the gradiant is

$$\vec{u} = \frac{(\vec{\nabla} f)_{(3,4)}}{|(\vec{\nabla} f)_{(3,4)}|} = \frac{-6\vec{i} - 8\vec{j}}{10} = -\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j} .$$

The derivative in this direction $\vec{u}$ is

$$(D_{\vec{u}} f)_{(3,4)} = |(\vec{\nabla} f)_{(3,4)}| \cos 0 = |(\vec{\nabla} f)_{(3,4)}|$$

$= 10$.

c) The derivative of f is zero in the directions perpendicular to $\vec{\nabla} f$. We can obtain these directions by interchanging the components of $\vec{u} = \dfrac{(\vec{\nabla} f)_{(3,4)}}{|(\vec{\nabla} f)_{(3,4)}|}$ and changing the sign of one of the new components.

$\therefore$ The results are $\vec{n}_1 = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$ and

$\vec{n}_2 = -\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$ (where $\vec{n}_2 = -\vec{n}_1$), at each of which the derivative is zero.

§5: Tangent Plane and Normal Line

Proposition 5.1: If $f(x,y,z)$ has continuous partial derivatives at a point $P_0(x_0,y_0,z_0)$ on the level surface $f(x,y,z)=c$ (where c is a constant). Then the tangent plane to the level surface $f(x,y,z)=c$ is the plane through P_0 normal to $\vec{\nabla} f$ at P_0 whose equation is

$$f_x(x_0,y_0,z_0)\cdot(x-x_0)+f_y(x_0,y_0,z_0)\cdot(y-y_0)$$

$$+f_z(x_0,y_0,z_0)\cdot(z-z_0)=0 \quad ,$$

and the normal line to the level surface $f(x,y,z)=c$ is the line perpendicular to the tangent plane and parallel to $\vec{\nabla} f$ at P_0 whose equations are

$$x=x_0+f_x(x_0,y_0,z_0)t \quad ,$$

$$y=y_0+f_y(x_0,y_0,z_0)t \quad ,$$

$$z=z_0+f_z(x_0,y_0,z_0)t \quad .$$

If none of $f_x(x_0,y_0,z_0)$, $f_y(x_0,y_0,z_0)$, $f_z(x_0,y_0,z_0)$ is zero, then the normal line is also given by the equations

$$\frac{x-x_0}{f_x(x_0,y_0,z_0)}=\frac{y-y_0}{f_y(x_0,y_0,z_0)}=\frac{z-z_0}{f_z(x_0,y_0,z_0)}$$

Example 5.2: Find the tangent plane and the normal line to the surface $x^2+xyz-z^2=1$ at the point $P_0(1,1,1)$.

Solution:

$$f_x(1,1,1)=f_x\Big|_{(1,1,1)}=(2x+yz)\Big|_{(1,1,1)}=3,$$

$$f_y(1,1,1)=f_y\Big|_{(1,1,1)}=xz\Big|_{(1,1,1)}=1,$$

$$f_z(1,1,1)=f_z\Big|_{(1,1,1)}=(xy-2z)\Big|_{(1,1,1)}=-1$$

Thus the equation of tangent plane to the surface $x^2+xyz-z^2=1$ at the point $P_0(1,1,1)$ is

$3(x-1)+(y-1)-(z-1)=0$ or

$3x+y-z=3$.

The equations of the normal line to the surface $x^2+xyz-z^2=1$ at the point $P_0(1,1,1)$ are

$x=1+3t$, $y=1+t$, $z=1-t$ or

$$\frac{x-1}{3}=\frac{y-1}{1}=\frac{z-1}{-1}$$ or

$x-1=3y-3=3-3z$.

Example 5.3: Find the tangent plane to the sphere $x^2+y^2+z^2=4$ at the point $(-1,1,\sqrt{2})$.

Solution: $f(x,y,z)=x^2+y^2+z^2$

$$f_x\Big|_{(-1,1,\sqrt{2})} = 2x\Big|_{(-1,1,\sqrt{2})} = -2 ,$$

$$f_y\Big|_{(-1,1,\sqrt{2})} = 2y\Big|_{(-1,1,\sqrt{2})} = 2 ,$$

$$f_z\Big|_{(-1,1,\sqrt{2})} = 2z\Big|_{(-1,1,\sqrt{2})} = 2\sqrt{2} .$$

$\therefore$ The equation of the tangent plane to the sphere $x^2+y^2+z^2=4$ at the point $(-1,1,\sqrt{2})$ is

$-2(x+1)+2(y-1)+2\sqrt{2}(z-\sqrt{2})=0$

$\Rightarrow -2x-2+2y-2+2\sqrt{2}\,z-4=0$

$\Rightarrow -2x+2y+2\sqrt{2}\,z=8$

$\Rightarrow -x+y+\sqrt{2}\,z=4 .$

Example 5.4: Find the tangent plane and the normal line to the surface $x^2+xyz-z^3=1$ at the point $P_0(1,1,1)$.

Solution: $f_x\Big|_{(1,1,1)} = (2x+yz)\Big|_{(1,1,1)} = 3$

$$f_y\Big|_{(1,1,1)} = xz\Big|_{(1,1,1)} = 1$$

$$f_z\Big|_{(1,1,1)} = (xy-3z^2)\Big|_{(1,1,1)} = -2$$

$\therefore$ The equation of the tangent plane to the surface at $P_0(1,1,1)$ is

$3(x-1)+1(y-1)-2(z-1)=0 \Rightarrow 3x+y-2z=2 .$

The equation of the normal line is

$$\frac{x-1}{3}=\frac{y-1}{1}=\frac{z-1}{-2} .$$