Solution: $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}$

$= -(v \sin uv) \cdot yz + (-u \sin uv) \cdot 2ax$

$= -a(x^2+y^2)(\sin(ax^3yz + axy^3z)) \cdot yz$

$-2ax^2yz(\sin(ax^3yz + axy^3z))$

Then $\frac{\partial w}{\partial x} = -a(1+1)(\sin 2a) \cdot 1$

$-2a(\sin 2a) = -4a \sin 2a$, at the point $(x,y,z) = (1,1,1)$.

Exercises 3.29:

1. Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos 2t$, $y = \sin 3t$, $z = 5t$. What is the derivative's value at $t = 0$?

2. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$ and then find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u,v) = (\frac{1}{2}, 1)$.

3. Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ at the point $(x,y,z) = (\sqrt{3}, 2, 1)$, if $u = \frac{p-q}{q-r}$, $p = x + y + z$, $q = x - y + z$, $r = x + y - z$.

Example 3.30: Find $\frac{\partial w}{\partial x}$ if $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$,

1. if x and y are independent variables,
2. if x and z are independent variables.

Solution:

1. With x and y are independent variables we get that

$$w = x^2 + y^2 + z^2 = x^2 + y^2 + (x^2 + y^2)^2$$

$$= x^2 + y^2 + x^4 + 2x^2y^2 + y^4.$$

Thus $\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2$.

2. With x and z are independent variables we will have $y^2 = z - x^2$, and we get that

$$w = x^2 + y^2 + z^2 = \cancel{x^2} + z - \cancel{x^2} + z^2 = z + z^2$$

Thus $\frac{\partial w}{\partial x} = 0$.

Remark 3.31: 1. In Example 3.30(1) we have $\frac{\partial w}{\partial y} = 2y + 4x^2y + 4y^3$.

2. In Example 3.30(2) we have

$$\frac{\partial w}{\partial z} = 1 + 2z.$$

Remark 3.32: To show what variables are assumed to be independent variables in calculating a derivative, we can use the following notations:

$\left(\frac{\partial w}{\partial x}\right)_y$ means $\frac{\partial w}{\partial x}$ with x and y independent,

$\left(\frac{\partial w}{\partial x}\right)_z$ means $\frac{\partial w}{\partial x}$ with x and z independent,

$\left(\frac{\partial w}{\partial y}\right)_{x,t}$ means $\frac{\partial w}{\partial y}$ with y, x, and t independent.

Example 3.33: If $w = x^2 + y - z + \sin t$ and $x + y = t$, then find

(a) $\left(\frac{\partial w}{\partial x}\right)_{y,z}$, (b) $\left(\frac{\partial w}{\partial x}\right)_{z,t}$,

(c) $\left(\frac{\partial w}{\partial y}\right)_{z,t}$, (d) $\left(\frac{\partial w}{\partial z}\right)_{x,t}$,

(e) $\left(\frac{\partial w}{\partial t}\right)_{x,z}$, (f) $\left(\frac{\partial w}{\partial t}\right)_{y,z}$.

Solution:

(a) With x, y, z are independent, we have $t = x + y$, which implies that $w = x^2 + y - z + \sin(x+y)$.

Thus $\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x + \cos(x+y)$.

(b) With x, z, t are independent, we have
$y = t - x$, which implies that
$w = x^2 + (t - x) - z + \sin t$.

Thus $\left(\frac{\partial w}{\partial x}\right)_{z,t} = 2x - 1$.

(c) With y, z, t are independent, we have
$x = t - y$, which implies that
$w = (t - y)^2 + y - z + \sin t$.

Thus $\left(\frac{\partial w}{\partial y}\right)_{z,t} = 2(t-y)\cdot(-1) + 1 = 2y - 2t + 1$.

(d) With z, x, t are independent, we have
$y = t - x$, which implies that
$w = x^2 + (t - x) - z + \sin t$.

Thus $\left(\frac{\partial w}{\partial z}\right)_{x,t} = -1$.

(e) With t, x, z are independent, we have
$y = t - x$, which implies that
$w = x^2 + (t - x) - z + \sin t$.

Thus $\left(\frac{\partial w}{\partial t}\right)_{x,z} = 1 + \cos t$.

(f) Exercise.

§4: Gradient and Directional Derivative

Definition 4.1: If the partial derivatives of $f(x,y,z)$ are defined at $P_0(x_0,y_0,z_0)$, then the gradient of f (or gradient vector of f) at P_0 is the vector

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \quad \text{or}$$

$$\vec{\nabla} f = f_x\,\vec{i} + f_y\,\vec{j} + f_z\,\vec{k}$$

obtained by evaluating the partial derivatives of f at P_0.

Remark 4.2: The symbol $\vec{\nabla} f$ may be read "grad f" or "gradient of f" or "del f".

Definition 4.3: If $f(x,y,z)$ has continuous partial derivatives at $P_0(x_0,y_0,z_0)$ and $\vec{u}$ is a unit vector, then the derivative of f at P_0 in the direction of $\vec{u}$ is the number

$$(D_{\vec{u}} f)_{P_0} = (\vec{\nabla} f)_{P_0} \cdot \vec{u}$$

which is the scalar product of $\vec{u}$ and the gradiant of f at P_0.

Remarks 4.4:

1. Another notation for the gradiant of f is

grad f .

2. Another notation for the directional derivative is

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0}.$$

Example 4.5: Find the directional derivative of $f(x,y) = x^2+xy+y^2$ at $P_0(1,2)$ in the direction of $\vec{A} = 2\vec{i}+3\vec{j}$.

Solution:

$$(D_{\vec{u}}f)_{P_0} = (\vec{\nabla}f)_{P_0} \cdot \vec{u}.$$

$$\vec{\nabla}f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$= (2x+y)\vec{i} + (x+2y)\vec{j}.$$

Thus $(\vec{\nabla}f)_{P_0} = 4\vec{i}+5\vec{j}$.

The unit vector in the direction of $\vec{A}$ is

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\vec{i}+3\vec{j}}{\sqrt{4+9}} = \frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j}$$

$$\therefore (D_{\vec{u}}f)_{P_0} = (4\vec{i}+5\vec{j})\cdot\left(\frac{2}{\sqrt{13}}\vec{i}+\frac{3}{\sqrt{13}}\vec{j}\right)$$

$$= \frac{8}{\sqrt{13}} + \frac{15}{\sqrt{13}} = \frac{23}{\sqrt{13}}.$$

Example 4.6: Find the directional derivative of $f(x,y,z) = e^x \cos(y+z)$ at the point $P_0(0,0,0)$ in the direction $\vec{A} = 2\vec{i} + \vec{j} - 2\vec{k}$.

Solution:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= (e^x \cos(y+z))\vec{i} + (-e^x \sin(y+z))\vec{j} + (-e^x \sin(y+z))\vec{k}$$

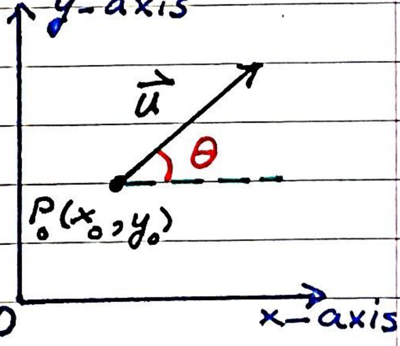
$$(\vec{\nabla} f)_{P_0} = 1.\vec{i} + 0.\vec{j} + 0.\vec{k} = \vec{i}$$

The unit vector in the direction of $\vec{A}$ is

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{4+1+4}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$$

$$\therefore (D_{\vec{u}} f)_{P_0} = \vec{i}.\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}\right) = \frac{2}{3}.$$

Remark 4.7: If f is a function of two variables x and y, and let $\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ be a unit vector with its initial point $P_0(x_0,y_0)$ and making an angle θ with the x-axis, then the directional derivative of f at $P_0(x_0,y_0)$ in the direction $\vec{u}$ can be written as



$$(D_{\vec{u}} f)_{P_0} = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta$$

Example 4.8: Find the directional derivative of $f(x,y)=x^2-3xy+2y^2$ at $P_0(-1,2)$ in the directions:

a) $\theta=\frac{\pi}{6}$

b) $\theta=\frac{\pi}{4}$

c) $\theta=2\pi$.

Solution:

$$f_x=2x-3y \Rightarrow f_x(-1,2)=2(-1)-3(2)$$
$$=-2-6=-8$$
$$f_y=-3x+4y \Rightarrow f_y(-1,2)=-3(-1)+4(2)$$
$$=3+8=11$$

a) When $\theta=\frac{\pi}{6}$, we have

$$(D_{\vec{u}}f)_{P_0}=-8\cos\frac{\pi}{6}+11\sin\frac{\pi}{6}$$
$$=-8\left(\frac{\sqrt{3}}{2}\right)+11\left(\frac{1}{2}\right)=\frac{-8\sqrt{3}+11}{2}.$$

b) When $\theta=\frac{\pi}{4}$, we have

$$(D_{\vec{u}}f)_{P_0}=-8\cos\frac{\pi}{4}+11\sin\frac{\pi}{4}$$
$$=-8\left(\frac{1}{\sqrt{2}}\right)+11\left(\frac{1}{\sqrt{2}}\right)=\frac{3}{\sqrt{2}}.$$

c) When $\theta=2\pi$, we have

$$(D_{\vec{u}}f)_{P_0}=-8\cos 2\pi+11\sin 2\pi$$
$$=-8(1)+11(0)=-8.$$