$$\therefore \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= (2x+5y^2)\cdot 2 + (10xy-z)\cdot\frac{1}{r} + (-y)\cdot e^{r+s}$$

$$= 4(2r+s^2) + 10(s+\ln r)^2 + \frac{10(2r+s^2)(s+\ln r)}{r}$$

$$-\frac{e^{r+s}}{r} - (s+\ln r)e^{r+s} \quad \text{, and}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= (2x+5y^2)\cdot 2s + (10xy-z)\cdot 1 + (-y)\cdot e^{r+s}$$

$$= (4r+2s^2)\cdot 2s + 10(s+\ln r)^2\cdot s + 10(2r+s^2)(s+\ln r)$$
$$- e^{r+s} - (s+\ln r)e^{r+s}$$

Remark 3.24: If f is a function of two variables x and y instead of three, then each equation of chain rules in theorem 3.22 becomes one term shorter, and the chain rules of $w=f(x(r,s),y(r,s))$ will be as follows:

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} \quad \text{, and}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

Example 3.25: Find $\frac{\partial w}{\partial s}$, $\frac{\partial w}{\partial t}$ as functions of s and t, if $w=f(x,y)=\frac{x}{\sqrt{x^2+y^2}}$, $x=2s-t$, $y=s+4t$.

Solution: $w = f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}} = x\cdot(x^2+y^2)^{-\frac{1}{2}}$.

$$\frac{\partial f}{\partial x} = x\cdot\left(-\tfrac{1}{2}\right)\cdot(x^2+y^2)^{-\frac{3}{2}}\cdot 2x + (x^2+y^2)^{-\frac{1}{2}}$$

$$= -x^2\cdot(x^2+y^2)^{-\frac{3}{2}} + (x^2+y^2)^{-\frac{1}{2}},$$

$$\frac{\partial f}{\partial y} = x\cdot\left(-\tfrac{1}{2}\right)\cdot(x^2+y^2)^{-\frac{3}{2}}\cdot 2y + (x^2+y^2)^{-\frac{1}{2}}\cdot 0$$

$$= -xy(x^2+y^2)^{-\frac{3}{2}},$$

$$\frac{\partial x}{\partial s} = 2, \quad \frac{\partial y}{\partial s} = 1, \quad \frac{\partial x}{\partial t} = -1, \quad \frac{\partial y}{\partial t} = 4.$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \left(-x^2\cdot(x^2+y^2)^{-\frac{3}{2}} + (x^2+y^2)^{-\frac{1}{2}}\right)\cdot 2$$

$$+ \left(-xy(x^2+y^2)^{-\frac{3}{2}}\right)\cdot 1$$

$$= -2(2s-t)^2\cdot\left((2s-t)^2+(s+4t)^2\right)^{-\frac{3}{2}}$$

$$+ 2\cdot\left((2s-t)^2+(s+4t)^2\right)^{-\frac{1}{2}}$$

$$- (2s-t)\cdot(s+4t)\left((2s-t)^2+(s+4t)^2\right)^{-\frac{3}{2}}$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \left(-x^2\cdot(x^2+y^2)^{-\frac{3}{2}} + (x^2+y^2)^{-\frac{1}{2}}\right)\cdot(-1)$$

$$+ \left(-xy(x^2+y^2)^{-\frac{3}{2}}\right)\cdot 4$$

$$= (2s-t)^2\cdot\left((2s-t)^2+(s+4t)^2\right)^{-\frac{3}{2}} - \left((2s-t)^2+(s+4t)^2\right)^{-\frac{1}{2}}$$

$$-4(2s-t)(s+4t)\left((2s-t)^2+(s+4t)^2\right)^{-\frac{3}{2}}.$$

Remark 3.26: If $w=f(x)$ is a function of x alone and x is a function of r and s (i.e. $x=x(r,s)$), then

$$\frac{\partial w}{\partial r}=\frac{dw}{dx}\cdot\frac{\partial x}{\partial r} \quad \text{and}$$
$$\frac{\partial w}{\partial s}=\frac{dw}{dx}\cdot\frac{\partial x}{\partial s}$$

Example 3.27: If a, b are constants and $w=f(u)$ is differentiable w.r.t. $u=ax+by$. Show that $a\cdot\frac{\partial w}{\partial y}=b\cdot\frac{\partial w}{\partial x}$.

Solution: From the chain rule we have

$$\frac{\partial w}{\partial x}=\frac{df}{du}\cdot\frac{\partial u}{\partial x}=\frac{df}{du}\cdot a \quad \text{and}$$
$$\frac{\partial w}{\partial y}=\frac{df}{du}\cdot\frac{\partial u}{\partial y}=\frac{df}{du}\cdot b$$

$$\therefore\ a\cdot\frac{\partial w}{\partial y}=a\cdot\frac{df}{du}\cdot b=ab\cdot\frac{df}{du} \quad \text{and}$$
$$b\cdot\frac{\partial w}{\partial x}=b\cdot\frac{df}{du}\cdot a=ab\cdot\frac{df}{du}.$$

Thus $a\cdot\frac{\partial w}{\partial y}=b\cdot\frac{\partial w}{\partial x}$.

Exercises 3.28: 1. By using the chain rule, find $\frac{du}{dt}$ if $u=3xy+e^{xy^2}$, $x=\tan 4t^2$, $y=\sin t\cos^{-1}t$, where $-1 < t < 1$.

Solution: $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$= (3y + y^2 e^{xy^2}).((\sec^2 4t^2).8t) +$

$(3x + 2xy e^{xy^2}).(\sin t.(-\frac{1}{\sqrt{1-t^2}}) + \cos^{-1} t.\cos t)$

$= \left(3\sin t\cos^{-1} t + (\sin t\cos^{-1} t)^2 . e^{\tan 4t^2.(\sin t\cos^{-1} t)^2}\right)$

$.8t\sec^2 4t^2 + (3\tan 4t^2 + 2\tan 4t^2 \sin t\cos^{-1} t$

$.e^{\tan 4t^2 (\sin t\cos^{-1} t)^2}).\left(\frac{-\sin t}{\sqrt{1-t^2}} + \cos t.\cos^{-1} t\right).$

2. By using the chain rule, find $\frac{dw}{dt}$ if

$w = x^4 + 2xy - 5y^2$, $x = \cosh t$, $y = \tan^{-1} t + e^t$.

Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$

$= (4x^3 + 2y).\sinh t + (2x - 10y).(\frac{1}{1+t^2} + e^t)$

$= (4\cosh^3 t + 2\tan^{-1} t + 2e^t).\sinh t$

$+ (2\cosh t - 10\tan^{-1} t - 10e^t)(\frac{1}{1+t^2} + e^t)$

3. By using the chain rule, find $\frac{dz}{d\theta}$ if

$z = \sqrt{x^2 + y^2}$, $x = \sin^2\theta$, $y = \ln(\cos\theta)$, where

$\cos\theta > 0$.

Solution: $\frac{dz}{d\theta} = \frac{\partial z}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial z}{\partial y} \cdot \frac{dy}{d\theta}$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \cdot 2\sin\theta . \cos\theta + \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} \cdot \frac{-\sin\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta . 2\sin\theta . \cos\theta}{\sqrt{\sin^4\theta + (\ln(\cos\theta))^2}} - \frac{\ln(\cos\theta) . \tan\theta}{\sqrt{\sin^4\theta + (\ln(\cos\theta))^2}}$$

$$= \frac{2\sin^3\theta . \cos\theta - \tan\theta . \ln(\cos\theta)}{\sqrt{\sin^4\theta + (\ln(\cos\theta))^2}} .$$

4. Find $\frac{\partial z}{\partial u}$ when $u=0$ and $v=1$, if

$z = \sin xy + x \sin y$, $x = u^2v^2$, $y = \frac{u+v}{v^2}$.

Solution: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$

$$= (y\cos xy + \sin y) . 2uv^2 + (x\cos xy + x\cos y) . \frac{1}{v^2}$$

$$= \left(\frac{u+v}{v^2} . \cos(u^3 + u^2v) + \sin\left(\frac{u+v}{v^2}\right)\right) . 2uv^2$$

$$+ \left(u^2v^2\cos(u^3+u^2v) + u^2v^2\cos\left(\frac{u+v}{v^2}\right)\right) . \frac{1}{v^2}$$

Then $\frac{\partial z}{\partial u} = (1.\cos 0 + \sin 1) . 0 + 0 = 0$, when

$u=0$ and $v=1$.

5. Find $\frac{\partial w}{\partial x}$ at the point $(x,y,z) = (1,1,1)$ if

$w = \cos uv$, $u = xyz$, $v = a(x^2+y^2)$.