Example 3.15: Determine whether the function $f(x,y) = \sin^{-1}\left(\frac{y}{x}\right)$ where $\left|\frac{y}{x}\right| < 1$, is harmonic or not.

Solution:

$$f_x = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2\sqrt{1-\frac{y^2}{x^2}}}$$

$$= \frac{-y}{\sqrt{x^4-x^2y^2}} = -y\,(x^4-x^2y^2)^{-\frac{1}{2}}.$$

$$f_{xx} = \frac{1}{2}y\,(x^4-x^2y^2)^{-\frac{3}{2}} \cdot (4x^3-2xy^2).$$

$$f_y = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \frac{1}{x} = \frac{1}{x\sqrt{1-\frac{y^2}{x^2}}} = \frac{1}{\sqrt{x^2-y^2}}$$

$$= (x^2-y^2)^{-\frac{1}{2}}$$

$$f_{yy} = -\frac{1}{2}(x^2-y^2)^{-\frac{3}{2}} \cdot (-2y) = y \cdot (x^2-y^2)^{-\frac{3}{2}}.$$

$$f_{xx} + f_{yy} = \frac{1}{2}y\,(x^4-x^2y^2)^{-\frac{3}{2}} \cdot (4x^3-2xy^2)$$

$$+ y \cdot (x^2-y^2)^{-\frac{3}{2}} \neq 0.$$

$\therefore$ f is not harmonic

Exercise 3.16: Determine whether each of the following functions is harmonic or not:

1. $z = f(x,y) = \cos^{-1}\left(\frac{y}{x}\right)$ where $\left|\frac{y}{x}\right| < 1$

   (Hint: $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$, $|u| < 1$).

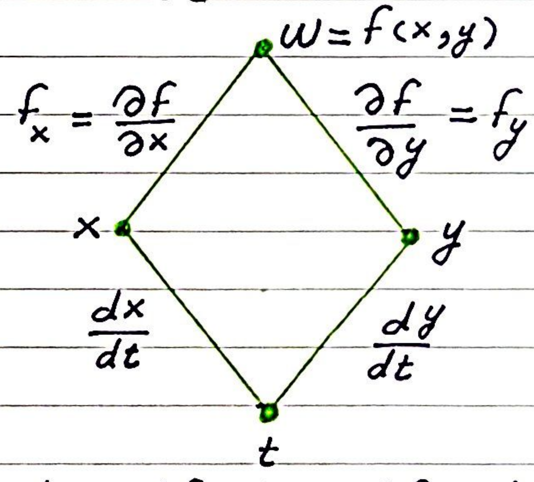
2. $f(x,y) = \ln(x^2+y^2)^2$.

Theorem 3.17 (Chain Rule for Functions of Two Independent Variables): If $w = f(x,y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite function $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} \qquad \text{or}$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

To find $\frac{dw}{dt}$ by the above chain rule you can use the following tree diagram by finding the product of the derivatives on each route starting from w ending at t, and then add these products to get $\frac{dw}{dt}$.



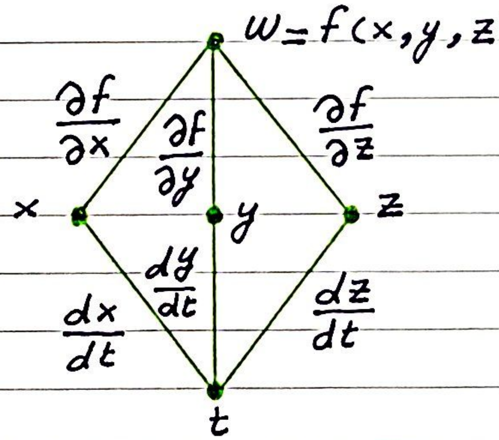
$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Theorem 3.18 (Chain Rule for Functions of Three Independent Variables): If $w=f(x,y,z)$ has continuous partial derivatives f_x, f_y, and f_z and if $x=x(t)$, $y=y(t)$, $z=z(t)$ are differentiable functions of t, then the composite function $w=f(x(t),y(t),z(t))$ is a differentiable function of t and

$$\frac{df}{dt}=f_x\cdot\frac{dx}{dt}+f_y\cdot\frac{dy}{dt}+f_z\cdot\frac{dz}{dt} \quad \text{or}$$

$$\frac{dw}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}.$$

To find $\frac{dw}{dt}$ by the above chain rule you can use the following tree diagram by finding the product of the derivatives on each route starting from w ending at t, and then add these products to get $\frac{dw}{dt}$.



$$\frac{dw}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

Example 3.19: Use the chain rule to find the derivative of $f(x,y) = x^5 + 6y^2$ w.r.t. t along the path $x = \ln t$, $y = e^t$.

Solution: $f_x = 5x^4$, $f_y = 12y$, $\frac{dx}{dt} = \frac{1}{t}$, $\frac{dy}{dt} = e^t$.

$$\therefore \frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} = 5x^4 \cdot \frac{1}{t} + 12y \cdot e^t$$

$$= 5(\ln t)^4 \cdot \frac{1}{t} + 12e^t \cdot e^t = \frac{5(\ln t)^4}{t} + 12e^{2t}.$$

Example 3.20: Use the chain rule to find the derivative of $w = x^3 - 3y + 5z$, when $x = t^2$, $y = e^t$, $z = \cos t$.

Solution: $w_x = 3x^2$, $w_y = -3$, $w_z = 5$, $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = e^t$, $\frac{dz}{dt} = -\sin t$.

$$\therefore \frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_y \cdot \frac{dy}{dt} + w_z \cdot \frac{dz}{dt}$$

$$= 3x^2 \cdot 2t + (-3) \cdot e^t + 5 \cdot (-\sin t)$$

$$= 3(t^2)^2 \cdot 2t - 3e^t - 5\sin t$$

$$= 6t^5 - 3e^t - 5\sin t$$

To check our answer we have

$$w = x^3 - 3y + 5z = (t^2)^3 - 3e^t + 5\cos t$$

$$= t^6 - 3e^t + 5\cos t$$

$$\therefore \frac{dw}{dt} = 6t^5 - 3e^t - 5\sin t.$$

Exercise 3.21: Use the chain rule to find the derivative of $z = \sin y + x e^{x}$ when $x = \ln(2t+10)$, $y = 5t+12$.

Theorem 3.22 (Chain Rules for Functions Defined on Surfaces):
If $w = f(x,y,z)$ is a function of x, y, z, and $x = x(r,s)$, $y = y(r,s)$, $z = z(r,s)$ are functions of r, s, and the functions f, x, y, z have continuous partial derivatives, then $w = f(x(r,s), y(r,s), z(r,s))$ is a function of r and s, and the partial derivatives of w w.r.t. r and s exist which are given by the following equations:

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Example 3.23: Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ of the function $w = f(x,y,z) = x^2 + 5xy^2 - zy$, if $x = 2r + s^2$, $y = s + \ln r$, $z = e^{r+s}$.

Solution: $\frac{\partial f}{\partial x} = 2x + 5y^2$, $\frac{\partial f}{\partial y} = 10xy - z$,

$\frac{\partial f}{\partial z} = -y$, $\frac{\partial x}{\partial r} = 2$, $\frac{\partial y}{\partial r} = \frac{1}{r}$, $\frac{\partial z}{\partial r} = e^{r+s}$,

$\frac{\partial x}{\partial s} = 2s$, $\frac{\partial y}{\partial s} = 1$, $\frac{\partial z}{\partial s} = e^{r+s}$.