§3: Partial Derivative

Definition 3.1: Let $z = f(x,y)$ be a function of two independent variables x and y.

1. If y is fixed, then f will be a function of one variable x, then we can derive $z = f(x, y_0)$ (y_0 is the fixed value of y) with respect to (w.r.t.) x, and this derivative is called the partial derivative of f w.r.t. x and denoted by f_x or $\frac{\partial f}{\partial x}$, hence f_x is a function of x and its value at (x_0, y_0) is $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$, where
$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$
2. If x is fixed, then f will be a function of one variable y, then we can derive $z = f(x_0, y)$ (x_0 is the fixed value of x) w.r.t. y, and this derivative is called the partial derivative of f w.r.t. y and denoted by f_y or $\frac{\partial f}{\partial y}$, hence f_y is a function of y and its value at (x_0, y_0) is $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$, where
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

Example 3.2: Find f_x, f_y, $f_x(1,2)$, $f_y(1,2)$ of each of the following functions:

1. $f(x,y) = 5xy - 7x^2 - y^2 - 3x - 6y - 2$,

2. $f(x,y)=\sqrt{9-x^2-y^2}$

Solution:

1. $f_x = 5y-14x-3$,
$f_y = 5x-2y-6$,

$f_x(1,2)=5(2)-14(1)-3=10-14-3=-7$,
$f_y(1,2)=5(1)-2(2)-6=5-4-6=-5$.

2. $f(x,y)=\sqrt{9-x^2-y^2}=(9-x^2-y^2)^{\frac{1}{2}}$

$$f_x=\frac{1}{2}(9-x^2-y^2)^{-\frac{1}{2}}.(-2x)=\frac{-x}{\sqrt{9-x^2-y^2}}$$

$$f_y=\frac{1}{2}(9-x^2-y^2)^{-\frac{1}{2}}.(-2y)=\frac{-y}{\sqrt{9-x^2-y^2}}$$

$$f_x(1,2)=\frac{-1}{\sqrt{9-1-4}}=\frac{-1}{\sqrt{4}}=-\frac{1}{2}$$

$$f_y(1,2)=\frac{-2}{\sqrt{9-1-4}}=\frac{-2}{\sqrt{4}}=-1$$

Remark 3.3 (Partial derivatives of functions with more than two variables):

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives w.r.t. the taken one variable by treating all the other independent variables as constants. Thus, if we have $f(x,y,z,u,v)$ as a function of five independent variables x, y, z, u, v, then we will have five partial derivatives f_x, f_y, f_z, f_u, f_v defined as follows:

$$f_x(x_0, y_0, z_0, u_0, v_0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0, u_0, v_0)$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0, z_0, u_0, v_0) - f(x_0, y_0, z_0, u_0, v_0)}{\Delta x},$$

$$f_y(x_0, y_0, z_0, u_0, v_0) = \frac{\partial f}{\partial y}(x_0, y_0, z_0, u_0, v_0)$$

$$= \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y, z_0, u_0, v_0) - f(x_0, y_0, z_0, u_0, v_0)}{\Delta y},$$

$$f_z(x_0, y_0, z_0, u_0, v_0) = \frac{\partial f}{\partial z}(x_0, y_0, z_0, u_0, v_0)$$

$$= \lim_{\Delta z \to 0} \frac{f(x_0, y_0, z_0+\Delta z, u_0, v_0) - f(x_0, y_0, z_0, u_0, v_0)}{\Delta z},$$

$$f_u(x_0, y_0, z_0, u_0, v_0) = \frac{\partial f}{\partial u}(x_0, y_0, z_0, u_0, v_0)$$

$$= \lim_{\Delta u \to 0} \frac{f(x_0, y_0, z_0, u_0+\Delta u, v_0) - f(x_0, y_0, z_0, u_0, v_0)}{\Delta u},$$

$$f_v(x_0, y_0, z_0, u_0, v_0) = \frac{\partial f}{\partial v}(x_0, y_0, z_0, u_0, v_0)$$

$$= \lim_{\Delta v \to 0} \frac{f(x_0, y_0, z_0, u_0, v_0+\Delta v) - f(x_0, y_0, z_0, u_0, v_0)}{\Delta v}$$

Example 3.4: Find $f_x, f_y, f_z, f_x(1,3,2)$, $f_y(1,3,2)$, $f_z(1,3,2)$ of the function $f(x,y,z) = e^{xyz}$.

Solution: $f_x = e^{xyz}.(yz)$,

$$f_y = e^{xyz}.(xz),$$

$$f_z = e^{xyz}.(xy),$$

$f_x(1,3,2) = 6e^6$, $f_y(1,3,2) = 2e^6$,

$f_z(1,3,2) = 3e^6$.

Exercise 3.5: Find f_x , f_y , f_z , $f_x(2,4,1)$ $f_y(2,4,1)$, $f_z(2,4,1)$ of each of the following functions:

1. $f(x,y,z) = \dfrac{x+y+z}{xy+yz+xz}$,

2. $f(x,y,z) = x^2 e^{yz} + \cos z^2 + \tan\frac{x}{z}$.

Definition 3.6: The slope of the curve $z = f(x, y_0)$ at $x = x_0$ is defined to be the value of the partial derivative of f with respect to x at (x_0, y_0)

Definition 3.7: The tangent to the curve $z = f(x, y_0)$ at $x = x_0$ is defined to be the straight line in the plane $y = y_0$ that passes through the point $(x_0, y_0, f(x_0, y_0))$ with a slope equal the partial derivative of $f(x, y_0)$ with respect to x at (x_0, y_0).

Example 3.8: Find the tangent to the curve $z = f(x,y) = 10 - xy$ in the plane $y = 4$ at the point $(2,4,2)$.

Solution: $f_x = -y$ and $f_x(2,4) = -4$.
Therefore the tangent to the curve f in the plane $y = 4$ is $\dfrac{z - z_0}{x - x_0} = -4 \Rightarrow \dfrac{z-2}{x-2} = -4$ and $y = 4$

$\Rightarrow z-2=-4x+8$ and $y=4$
$\Rightarrow z+4x=10$ and $y=4$.

Definition 3.9: The slope of the curve $z=f(x_0,y)$ at $y=y_0$ is defined to be the value of the partial derivative of f with respect to y at (x_0,y_0).

Definition 3.10: The tangent to the curve $z=f(x_0,y)$ at $y=y_0$ is defined to be the straight line in the plane $x=x_0$ that passes through the point $(x_0,y_0,f(x_0,y_0))$ with a slope equal the partial derivative of $f(x_0,y)$ with respect to y at (x_0,y_0).

Example 3.11: Find the tangent to the curve $z=f(x,y)=10-xy$ in the plane $x=2$ at the point $(2,4,2)$.
Solution: $f_y=-x$ and $f_y(2,4)=-2$.
Therefore the tangent to the curve f in the plane $x=2$ is $\frac{z-z_0}{y-y_0}=-2 \Rightarrow \frac{z-2}{y-4}=-2$ and $x=2$

$\Rightarrow z-2=-2y+8$ and $x=2 \Rightarrow z+2y=10$ and $x=2$.

Remark 3.12: If f is a function of two variables x and y, then we have four second partial derivatives of f denoted as follows:

1. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}$,

2. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$,

3. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=f_{xy}$ (i.e. for f_{xy} we should differentiate f first with respect to x to obtain f_x, then we differentiate f_x with respect to y to obtain $f_{xy}=(f_x)_y$.

4. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{yx}$ (i.e. for f_{yx} we should differentiate f first with respect to y to obtain f_y, then we differentiate f_y with respect to x to obtain $f_{yx}=(f_y)_x$

Definition 3.13: A function $z=f(x,y)$ (of two independent variables x and y) is called harmonic if f satisfy Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$.

Example 3.14: Determine whether the following function is harmonic or not

$$f(x,y)=e^{-y}\cos x.$$

Solution: $f_x=-e^{-y}\sin x \Rightarrow f_{xx}=-e^{-y}\cos x$.

$$f_y=-e^{-y}\cos x \Rightarrow f_{yy}=e^{-y}\cos x.$$

Thus $f_{xx}+f_{yy}=-e^{-y}\cos x+e^{-y}\cos x=0$.

Therefore f is a harmonic function.