Then the $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ is not exist.

11) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2}$ is not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right)$$

$= \lim\limits_{x\to 0} \frac{x^2}{x^2} = 1$ , and

$$\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right)$$

$= \lim\limits_{y\to 0} \frac{y^2}{y^2} = 1$.

Although $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right) = \lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right)$,

but $\lim\limits_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2}$ is not exist since

if we take the limit on the line $y=2x$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2+x^2+(2x)^2}{x^2+(2x)^2}$$

$= \lim\limits_{x\to 0} \frac{7x^2}{5x^2} = \frac{7}{5}$ which is not equal 1.

Example 2.7: Examine the limit of the function $f(x,y)=\dfrac{x^2y}{x^4+y^2}$, $(x,y)\neq(0,0)$, as $(x,y)\to(0,0)$ along the lines $y=mx$ $(m\neq 0)$ and along the parabola $y=x^2$. Does $f$ has a limit as $(x,y)\to(0,0)$?

Solution: Along the lines $y=mx$ $(m\neq 0)$

$$f(x,y)=f(x,mx)=\frac{x^2(mx)}{x^4+(mx)^2}=\frac{mx^3}{x^4+m^2x^2}$$

$$=\frac{mx\,(x^2)}{(x^2+m^2)(x^2)}=\frac{mx}{x^2+m^2}.$$

Therefore $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx\ (m\neq 0)}} f(x,y)=\lim_{x\to 0}\frac{mx}{x^2+m^2}=\frac{0}{m^2}=0.$

Along the parabola $y=x^2$

$$f(x,y)=f(x,x^2)=\frac{x^2.(x^2)}{x^4+(x^2)^2}=\frac{x^4}{2x^4}=\frac{1}{2}$$

Therefore $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^2}} f(x,y)=\lim_{x\to 0}\frac{1}{2}=\frac{1}{2}$

For $f$ to have a limit as $(x,y)\to(0,0)$, the limits along all paths of approach to $(0,0)$ must be the same.

Since we have two different limits on two different paths, then $f$ has no limit as $(x,y)\to(0,0)$.

Exercise 2.8: Does $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^4y}{x^6+y^3}$ exist or not?

**Remark 2.9:** For the limits of the functions of three or more variables we will apply theorems similar to theorems 2.4 and 2.5 .

**Example 2.10:** Find the following limits :

1) $$\lim_{(x,y,z)\to(2,4,3)} z = 3$$

2) $$\lim_{(x,y,z)\to(1,1,1)} \sqrt{x^2+y^2+z^2-1} = \sqrt{2}$$

3) $$\lim_{(x,y,z)\to(2,3,3)} \frac{xy^2-xz^2}{y-z}$$

$$= \lim_{(x,y,z)\to(2,3,3)} \frac{x(y^2-z^2)}{(y-z)}$$

$$= \lim_{(x,y,z)\to(2,3,3)} \frac{x(y+z)\cancel{(y-z)}}{\cancel{(y-z)}} = 12 .$$

**Definition 2.11:** A function $f(x,y)$ is said to be continuous at the point $(x_0,y_0)$ , if

1) $f$ is defined at $(x_0,y_0)$ ,

2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists , and

3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ .

## Remarks 2.12 :

1) If $f(x,y)$ and $g(x,y)$ are both continuous at a point $(x_0,y_0)$, then the sum $f+g$, difference $f-g$, product $f \cdot g$, and quotient $\frac{f}{g}$, $g \neq 0$ are continuous at $(x_0,y_0)$.

2) The polynomial functions of two variables are continuous everywhere.

3) The rational functions of two polynomial functions of two variables are continuous whenever the denominator is not zero.

4) If $f(z)$, $g(x,y)$ are continuous, then $f \circ g$ is continuous and $f \circ g(x,y) = f(g(x,y))$, where $z = g(x,y)$. i.e. if $z = g(x,y)$ is continuous at a point $(x_0,y_0)$ and $w = f(z)$ is continuous at $z_0 = g(x_0,y_0)$ then $w = f(g(x,y))$ is continuous at the point $(x_0,y_0)$.

## Examples 2.13 :

1) Let $f(x,y) = xy + x^2$ and $g(x,y) = xy^2$ which are continuous at the point $(1,2)$, then the functions $(f+g)(x,y) = xy + x^2 + xy^2$, $(f-g)(x,y) = xy + x^2 - xy^2$, $(f \cdot g)(x,y) = (xy + x^2)xy^2 = x^2y^3 + x^3y^2$, and $\left(\frac{f}{g}\right)(x,y) = \frac{xy + x^2}{xy^2}$ are all continuous at the point $(1,2)$.

2) The function $f(x,y)=x^6+x^2y^2+y^3+50$ is a continuous function everywhere.

3) The function $h(x,y)=\dfrac{2x+4y}{y^2-4x}$ is a continuous function everywhere except at the points on the parabola $y^2=4x$.

4) Let $f(z)=e^z$ and $g(x,y)=x-y$, g is continuous at the point $(4,1)$ and f is continuous at 3, then the function $(f\circ g)(x,y)=f(g(x,y))$

$=f(x-y)=e^{x-y}$ is continuous at the point $(4,1)$.

Example 2.14: Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x+y} & (x,y) \text{ not on the line } y=-x\\ 6 & (x,y) \text{ on the line } y=-x\end{cases}$$

is continuous at the point $(3,-3)$.

Solution:

1. $f(3,-3)=6$ is defined.
2. $$\lim_{\substack{(x,y)\to(3,-3)\\ y\neq -x}} \frac{x^2-y^2}{x+y} = \lim_{\substack{(x,y)\to(3,-3)\\ y\neq -x}} \frac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}}$$

$$= \lim_{\substack{(x,y)\to(3,-3)\\ y\neq -x}} (x-y) = 6$$

and $\lim\limits_{\substack{(x,y)\to(3,-3)\\ y=-x}} 6 = 6$ .

Therefore $\lim\limits_{(x,y)\to(3,-3)} f(x,y) = 6$ .

3. $\lim\limits_{(x,y)\to(3,-3)} f(x,y) = 6 = f(3,-3)$ .

Therefore $f$ is continuous at the point $(3,-3)$

Example 2.15: Prove that the function

$$f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

is continuous at every point.

Solution: Since the function $f(x,y)$

$$= \frac{x^4-y^4}{x^2+y^2} = \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = x^2-y^2$$

defined at any point $(x,y)\neq(0,0)$ , then $f$ is continuous at any point $(x,y)\neq(0,0)$ , because $x^2-y^2$ is a polynomial function of $x$ and $y$.

Now for the point $(0,0)$ , we have

1. $f$ is defined at $(0,0)$ and $f(0,0)=0$ ,
2. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} (x^2-y^2) = 0$ , and
3. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ .

Thus $f$ is continuous at $(0,0)$.

Therefore $f$ is continuous at every point.

Example 2.16:

Let $f(x,y)=\begin{cases} x^3+4y & \text{if } (x,y)\neq(1,1) \\ 0 & \text{if } (x,y)=(1,1) \end{cases}$.

Is $f$ continuous at $(1,1)$?

Solution: 1. $f$ is defined at $(1,1)$, and $f(1,1)=0$,

2. $\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} (x^3+4y) = 5$,

3. $f(1,1) \neq \lim_{(x,y)\to(1,1)} f(x,y)$

Therefore $f$ is not continuous at the point $(1,1)$.

Exercise 2.17: Find the following limits

1) $\lim_{(x,y,z)\to(0,1,1)} \dfrac{e^{x+y+z}}{z+\sin\sqrt{x^2y}}$

2) $\lim_{(x,y,z)\to(1,2,2)} \dfrac{\ln\sqrt{x^2+2y^2+z^2}}{xyz}$.

Exercise 2.18: Show that the function

$$f(x,y)=\begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

is continuous at every point except the point $(0,0)$.