§2: Limits and Continuity

Remark 2.1: If the values of a function $z=f(x,y)$ can be made as close as we like to a fixed number L by taking the point (x,y) close to the point (x_0,y_0), but not equal to (x_0,y_0), we say that L is the limit of f as (x,y) approaches (x_0,y_0), and we write it as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Also we can say that the limit of f as (x,y) approaches (x_0,y_0) equals L.

Definition 2.2: The limit of $f(x,y)$ as $(x,y)\to(x_0,y_0)$ is the number L if for any $\varepsilon>0$ there exists a $\delta>0$ such that for all $(x,y)\neq(x_0,y_0)$ in the domain of f we have

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y)-L| < \varepsilon.$$

The following definition is equivalent to the above definition 2.2.

Definition 2.3: The limit of $f(x,y)$ as $(x,y)\to(x_0,y_0)$ is the number L if for any $\varepsilon>0$ there exists a $\delta>0$ such that for all $(x,y)\neq(x_0,y_0)$ in the domain of f we have

$$|x-x_0|<\delta \text{ and } |y-y_0|<\delta \Rightarrow |f(x,y)-L|<\varepsilon.$$

Theorem 2.4 :

1) $\lim_{(x,y)\to(x_0,y_0)} x = x_0$,

2) $\lim_{(x,y)\to(x_0,y_0)} y = y_0$,

3) $\lim_{(x,y)\to(x_0,y_0)} k = k$, where k is any constant .

Theorem 2.5 : If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$, then

1) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)+g(x,y)] = L_1 + L_2$,

2) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)-g(x,y)] = L_1 - L_2$,

3) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\cdot g(x,y)] = L_1 \cdot L_2$,

4) $\lim_{(x,y)\to(x_0,y_0)} [k . f(x,y)] = k . L_1$ (k is any constant),

5) $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$.

Example 2.6: Find the following limits:

1) $\lim\limits_{(x,y)\to(1,1)} (x^2+y^2) = 1^2+1^2 = 2$.

2) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x^2-xy+10}{x^2y+2xy+y^2} = \dfrac{10}{1} = 10$.

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin x}{\sqrt{x^2+3}} = \dfrac{0}{\sqrt{3}} = 0$.

4) $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x+y} = e^{\ln 2} = 2$.

5) $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}} = \dfrac{0}{\sqrt{4}} = \dfrac{0}{2} = 0$.

6) $\lim\limits_{(x,y)\to(0,0)} \cos\left(\dfrac{x^2+y^2}{x+y+1}\right) = \cos 0 = 1$.

7) $\lim\limits_{(x,y)\to(1,1)} \cos\sqrt[3]{xy-1} = \cos 0 = 1$.

8) $\lim\limits_{(x,y)\to(3,4)} \dfrac{x^2-y^2}{x-y} = \lim\limits_{(x,y)\to(3,4)} \dfrac{\cancel{(x-y)}(x+y)}{\cancel{(x-y)}}$

$= \lim\limits_{(x,y)\to(3,4)} (x+y) = 3+4 = 7$.

9) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x^2-y^2} = \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x^2+y^2}{x^2-y^2}\right)$

$= \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$, and also we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2+y^2}{x^2-y^2}\right)$$

$$= \lim_{y\to 0} \frac{y^2}{-y^2} = -1 .$$

Since $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2+y^2}{x^2-y^2}\right) \neq \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2+y^2}{x^2-y^2}\right)$

Therefore $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$ does not exist.

10) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ is not exist since the limit on the straight line $y=2x$ which pass through the point $(0,0)$ is

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{x^2+y^2}{xy} = \lim_{x\to 0} \frac{x^2+(2x)^2}{2x^2}$$

$$= \lim_{x\to 0} \frac{5x^2}{2x^2} = \frac{5}{2},$$

while the limit on the straight line $y=3x$ which pass through the point $(0,0)$ is

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=3x}} \frac{x^2+y^2}{xy} = \lim_{x\to 0} \frac{x^2+(3x)^2}{3x^2} = \lim_{x\to 0} \frac{10x^2}{3x^2} = \frac{10}{3}$$

Since $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{x^2+y^2}{xy} \neq \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=3x}} \frac{x^2+y^2}{xy}$

Then the $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ is not exist.

11) Show that $\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2}$ is not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right)$$

$= \lim_{x\to 0} \frac{x^2}{x^2} = 1$, and

$$\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right)$$

$= \lim_{y\to 0} \frac{y^2}{y^2} = 1$.

Although $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right) = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy+x^2+y^2}{x^2+y^2}\right)$,

but $\lim_{(x,y)\to(0,0)} \frac{xy+x^2+y^2}{x^2+y^2}$ is not exist since

if we take the limit on the line $y=2x$, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{xy+x^2+y^2}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2+x^2+(2x)^2}{x^2+(2x)^2}$$

$= \lim_{x\to 0} \frac{7x^2}{5x^2} = \frac{7}{5}$ which is not equal 1.