CHAPTER FOUR

Interpolation

We use the interpolation with the following given data

i	0	1	2	3	...	n
x_i	x_0	x_1	x_2	x_3	...	x_n
y_i	y_0	y_1	y_2	y_3	...	y_n

and there is no formula for the function $y = f(x)$.

To find the value of the dependent variable y that correspond to any independent variable x, we use the interpolation method.

We use the given data to find the interpolating polynomial $P_n(x)$ which represent the function $y = f(x)$ in the polynomial form.

If the given data consist of $n+1$ distinct points, then the interpolating polynomial $P_n(x)$ will be of at most the nth degree, and $P_n(x)$ satisfies that $P_n(x_i) = y_i$, for each $i = 0, 1, 2, \ldots, n$, which means that the polynomial function $P_n(x)$

pass through all the points (x_i , y_i) , $i = 0, 1, 2, \ldots, n$.

* نستخدم الاستكمال مع البيانات المعطاة التالية

i	0	1	2	3	...	n
x_i	x_0	x_1	x_2	x_3	...	x_n
y_i	y_0	y_1	y_2	y_3	...	y_n

وليس هنالك صيغة رياضية متوفرة للدالة $y = f(x)$.

* لإيجاد قيمة المتغير المعتمد y الذي يقابل المتغير المستقل x فإن علينا استخدام طريقة الاستكمال.

* نستخدم البيانات المعطاة لإيجاد كثيرة حدود الاستكمال $P_n(x)$ والتي تمثل الدالة $y = f(x)$ بصيغة متعددة حدود.

* إذا كانت البيانات المعطاة تتضمن $n+1$ نقطة مختلفة فإننا سنحصل على دالة الاستكمال الكثيرة الحدود $P_n(x)$ من الدرجة n أو أقل وأن $P_n(x)$ تحقق العلاقة $P_n(x_i) = y_i$ لكل $i = 0, 1, 2, \ldots, n$

وهذا يعني أن الدالة $P_n(x)$ تمر من جميع النقاط المعطاة $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots (x_n, y_n)$.

Lagrange Interpolation:

The Lagrange interpolation polynomial function $P_n(x)$ is of degree at most n, which is given in the following formula $(*)$:

دالة الاستكمال الكثيرة الحدود للاكرانج $P_n(x)$ والتي هي من الدرجة n أو أقل معطاة بالصيغة التالية:

If the given data consist of $n+1$ distinct points, then

$$P_n(x) = L_0(x)y_0 + L_1(x)y_1 + \cdots + L_n(x)y_n$$

$$= \sum_{i=0}^{n} L_i(x)\, y_i \qquad \cdots (*)$$

where $L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \dfrac{x - x_j}{x_i - x_j}$

Now, if $n=1$ (which means that two points (x_0, y_0) and (x_1, y_1) are given), then

$$P_1(x) = L_0(x)y_0 + L_1(x)y_1 = \sum_{i=0}^{1} L_i(x)\, y_i$$

where $L_0(x) = \dfrac{x - x_1}{x_0 - x_1}$, and $L_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

and if $n=2$ (which means that three point (x_0, y_0), (x_1, y_1), and (x_2, y_2) are given), then

$$P_2(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$
$$= \sum_{i=0}^{2} L_i(x)y_i \quad \text{, where}$$

$$L_0(x) = \frac{x-x_1}{x_0-x_1} * \frac{x-x_2}{x_0-x_2} \quad ,$$

$$L_1(x) = \frac{x-x_0}{x_1-x_0} * \frac{x-x_2}{x_1-x_2} \quad \text{, and}$$

$$L_2(x) = \frac{x-x_0}{x_2-x_0} * \frac{x-x_1}{x_2-x_1} \quad .$$

Example 1: Use Lagrange interpolation to estimate $y(1.22)$ if $y(1.2) = 0.3849$, and $y(1.3) = 0.4032$.

Solution:

i	0	1
x_i	$x_0 = 1.2$	$x_1 = 1.3$
y_i	$y_0 = 0.3849$	$y_1 = 0.4032$

Since we have two given points, then $n=1$.

Thus $P_n(x) = P_1(x) = L_0(x)\,y_0 + L_1(x)\,y_1$

$$= \sum_{i=0}^{1} L_i(x)\, y_i \quad , \text{ where}$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 1.3}{1.2 - 1.3} = \frac{x - 1.3}{-0.1}$$

$$= -10x + 13 \quad , \text{ and}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 1.2}{1.3 - 1.2} = \frac{x - 1.2}{0.1}$$

$$= 10x - 12$$

$$\therefore P_1(x) = (-10x + 13) * (0.3849) + (10x - 12) * (0.4032)$$

$$= -3.849x + 5.0037 + 4.032x - 4.8384$$

$$= 0.183x + 0.1653$$

We use the above Lagrange interpolation polynomial function $P_1(x)$ to estimate the value of y at $x = 1.22$ as follows:

$$y(1.22) = P_1(1.22) = 0.183 * 1.22 + 0.1653$$

$$= 0.38856 \; .$$

Thus the interpolated point is $(x, y) = (1.22, 0.38856)$, which means that when $x = 1.22$, then $y = 0.38856$.

Example 2: Use Lagrange interpolation to estimate $y(3)$, if $y(2)=0.5$, $y(2.5)=0.4$, and $y(4)=0.25$.

Solution:

i	0	1	2
x_i	$x_0=2$	$x_1=2.5$	$x_2=4$
y_i	$y_0=0.5$	$y_1=0.4$	$y_2=0.25$

Since we have three given points, then $n=2$.

Thus $P_n(x)=P_2(x)=L_0(x)y_0+L_1(x)y_1+L_2(x)y_2$

$$=\sum_{i=0}^{2} L_i(x)y_i \quad \text{, where}$$

$$L_0(x)=\frac{x-x_1}{x_0-x_1}*\frac{x-x_2}{x_0-x_2}=\frac{x-2.5}{2-2.5}*\frac{x-4}{2-4}$$

$$=\frac{x-2.5}{-0.5}*\frac{x-4}{-2}=(x-2.5)(x-4)$$

$$=x^2-6.5x+10,$$

$$L_1(x)=\frac{x-x_0}{x_1-x_0}*\frac{x-x_2}{x_1-x_2}=\frac{x-2}{2.5-2}*\frac{x-4}{2.5-4}$$

$$=\frac{x-2}{0.5}*\frac{x-4}{-1.5}=\frac{x^2-6x+8}{-0.75}$$

$$=-\frac{4}{3}x^2+8x-\frac{32}{3},$$

$$L_2(x) = \frac{x - x_0}{x_2 - x_0} * \frac{x - x_1}{x_2 - x_1} = \frac{x-2}{4-2} * \frac{x-2.5}{4-2.5}$$

$$= \frac{x-2}{2} * \frac{x-2.5}{1.5} = \frac{x^2 - 4.5x + 5}{3}$$

$$= \frac{1}{3}x^2 - 1.5x + \frac{5}{3}$$

$$\therefore P_2(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$

$$= (x^2 - 6.5x + 10) * 0.5$$

$$+ \left(-\frac{4}{3}x^2 + 8x - \frac{32}{3}\right) * 0.4$$

$$+ \left(\frac{1}{3}x^2 - 1.5x + \frac{5}{3}\right) * 0.25$$

$$= 0.5x^2 - 3.25x + 5 - \frac{16}{30}x^2 + 3.2x - \frac{128}{30}$$

$$+ \frac{25}{300}x^2 - 0.375x + \frac{125}{300}$$

$$= 0.05x^2 - 0.425x + 1.15$$

We use the above Lagrange interpolation polynomial function to estimate the value of y at $x = 3$ as follows:

$$y(3) \approx P_2(3) = 0.05 * 3^2 - 0.425 * 3 + 1.15$$
$$= 0.325.$$

Thus the interpolated point is
$(x, y) = (3, 0.325)$.

Exercises :

1) Use Lagrange interpolation to estimate $y(4.3)$ if $y(2.2) = 0.451$, and $y(5.1) = 0.886$.

2) Use Lagrange interpolation to estimate $y(5)$ if $y(2) = 1$, $y(4) = 0.5$, $y(10) = 0.2$.