6) The fixed point iteration method:

This method is described as follows:

Algorithm Steps

Given $f(x)=0$

1) Find a and b such that $f(a).f(b)<0$.
2) Take $x_0=\frac{a+b}{2}$.
3) Write $f(x)=0$ in the form $x=g(x)$ in many different ways.
4) Find $g'(x)$ of each equation $x=g(x)$ which are found in (3)
5) Test whether $|g'(x_0)|<1$ or not to see if g is convergent or divergent.
   g is convergent if $|g'(x_0)|<1$ and
   g is divergent if $|g'(x_0)|\geq 1$.
6) Choose one of the convergent g.
7) Replace x by x_{i+1} and $g(x)$ by $g(x_i)$ in the equation $x=g(x)$.
8) Find the values of $x_1, x_2, x_3, \ldots$ and stop at x_{n+1} when $|x_{n+1}-x_n|<\epsilon$.

Example: By using the fixed point iteration method, find the value of a root of the equation $x^2+4x-3=0$ correct to three decimal places (i.e. $|x_{i+1}-x_i|<0.001$).

Solution:

Let $f(x)=x^2+4x-3$.

Since $f(0) = 0^2 + 4(0) - 3 = -3$ (−ve)
and $f(1) = 1^2 + 4 - 3 = 2$ (+ve)
which implies that $f(0) \cdot f(1) < 0$ and that a root of the equation $f(x) = 0$ lies between 0 and 1.
Thus we take $x_0 = \frac{0+1}{2} = 0.5$

a) $x^2 + 4x - 3 = 0 \Rightarrow x^2 = 3 - 4x$
$\Rightarrow x = \sqrt{3-4x}$
Thus $g(x) = \sqrt{3-4x} = (3-4x)^{\frac{1}{2}}$

and $g'(x) = \frac{1}{2}(3-4x)^{\frac{1}{2}-1} \cdot (-4) = \dfrac{-2}{\sqrt{3-4x}}$.

Hence $g'(x_0) = \dfrac{-2}{\sqrt{3-4(0.5)}} = -2$

and $|g'(x_0)| > 1$ (g is divergent).

b) $x^2 + 4x - 3 = 0 \Rightarrow 4x = 3 - x^2 \Rightarrow x = \dfrac{3-x^2}{4}$.
Thus $g(x) = \frac{3}{4} - \frac{1}{4}x^2$ and $g'(x) = -\frac{1}{2}x$.
Hence $g'(x_0) = -\frac{1}{2}(0.5) = -0.25$
and $|g'(x_0)| = 0.25 < 1$ (g is convergent)

c) $x^2 + 4x - 3 = 0 \Rightarrow x^2 + 4x = 3 \Rightarrow x(x+4) = 3$
$\Rightarrow x = \dfrac{3}{x+4}$.
Thus $g(x) = \dfrac{3}{x+4}$ and $g'(x) = \dfrac{-3}{(x+4)^2}$

Hence $g'(x_0) = \frac{-3}{(0.5+4)^2} = \frac{-3}{20.25} = -0.148148$

and $|g'(x_0)| = 0.148148 < 1$ (g is convergent)

d) $x^2+4x-3=0 \Rightarrow x(x+4)=3$

$\Rightarrow x+4 = \frac{3}{x} \Rightarrow x = \frac{3}{x} - 4$.

Thus $g(x) = \frac{3}{x} - 4$ and $g'(x) = \frac{-3}{x^2}$.

Hence $g'(x_0) = \frac{-3}{(0.5)^2} = -12$ and

$|g'(x_0)| = 12 > 1$ (g is divergent).

We select the formula $g(x) = \frac{3}{x+4}$ in (c)

Thus we have the formula $x_{i+1} = \frac{3}{x_i+4}$

Hence $x_1 = \frac{3}{0.5+4} = 0.6666666667$,

$x_2 = \frac{3}{0.6666666667+4} = 0.6428571429$

$x_3 = \frac{3}{0.6428571429+4} = 0.6461538461$

$x_4 = \frac{3}{0.6461538461+4} = 0.6456953642$

$|x_4 - x_3| = 0.0004584819 < 0.001$.

Therefore the root $\bar{x} \simeq 0.646$

Exercises:

1) By using the fixed point iteration method, find the value of a root of the equation $x^2+2x-1=0$ correct to three decimal places (i.e. $|x_{i+1}-x_i| < 0.001$).

2) By using the fixed point iteration method, find the value of a root of the equation $\sin x - x^2+1=0$ such that $|x_{i+1}-x_i| < 0.002$ by taking $x_0=1.5$.