the variable x, which is denoted by $f'(x), y', \frac{dy}{dx}, \frac{d}{dx}y, \frac{d}{dx}f(x)$.

Differentiation Rules:

Let $f(x)$ and $g(x)$ be two differentiable functions (in the interval under consideration), then

RULE 1 Constant Multiple Rule

If $f(x)$ is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(\,c\,f(x)\,) = c\,\frac{d}{dx}f(x)\,.$$

RULE 2 Derivative of the Sum

If $f(x)$ and $g(x)$ are differentiable functions of x, then their sum $f(x) + g(x)$ is differentiable, and

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

RULE 3 Derivative of the Difference

If $f(x)$ and $g(x)$ are differentiable functions of x, then their difference $f(x) - g(x)$ is differentiable, and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

RULE 4 Derivative of the Product

If $f(x)$ and $g(x)$ are differentiable functions of x, then their product $f(x).\,g(x)$ is differentiable, and

$$\frac{d}{dx}(f(x).\,g(x)) = f(x).\,\frac{d}{dx}g(x) + g(x).\,\frac{d}{dx}f(x)$$

RULE 5 Derivative of the Quotient

If $f(x)$ and $g(x)$ are differentiable functions of x and $g(x) \neq 0$, then the quotient $\frac{f(x)}{g(x)}$ is differentiable, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{g(x)^2}$$

Derivatives of Some Special Functions and the Chain Rule:

1) Derivatives of Some Algebraic Functions:

1) Derivative of a Constant Function

If $f(x) = c$, then $\frac{d}{dx} f(x) = \frac{d}{dx} c = 0$

Example 3.2.3 : If $f(x) = 12$, then $\frac{d}{dx} f(x) = \frac{d}{dx}(12) = 0$.

2) Derivatives of a Power Functions

$$\frac{d}{dx} x^n = n\, x^{n-1} \;,\; n \in Q$$

provided that $x \neq 0$ when n is negative .

Example 3.2.4 : Find f' for each of the following functions :

(*i*) $f(x) = x$, (*ii*) $f(x) = x^2$, (*iii*) $f(x) = x^{-3}$, (*iv*) $f(x) = x^{0.3}$

Solution:

(*i*) $f'(x) = x^{1-1} = x^0 = 1$

(*ii*) $f'(x) = 2x^{2-1} = 2x$

(*iii*) $f'(x) = -3x^{-3-1} = -3x^{-4}$

(*iv*) $f'(x) = 0.3x^{0.3-1} = 0.3\,x^{-0.7}$

Example 3.2.5 : Find f' for each of the following functions :

(*i*) $f(x)=\frac{1}{2}x$, (*ii*) $f(x)=9x^2$, (*iii*) $f(x)=4x^{-3}$, (*iv*) $f(x)=x^{2.5}$,

Solution:

(*i*) $f'(x)=\frac{1}{2}(1)=\frac{1}{2}$

(*ii*) $f'(x)=9(2x^{2-1}) = 18x$

(*iii*) $f'(x)=4((-3)x^{-3-1}) = -12x^{-4}$

(*iv*) $f'(x)=2.5x^{2.5-1}=2.5x^{1.5}$

Example 3.2.6 : Find f' for each of the following functions :

(*i*) $f(x)= x^2+5x^{-3}$, (*ii*) $f(x)=x^4-\frac{3}{5}x^2+7x-14$

Solution:

(*i*) $f'(x) = 2x - 15x^{-4}$

(*ii*) $f'(x)=4x^3-\frac{3(2)}{5}x + 7 - 0 = 4x^3-\frac{6}{5}x+7$

Example 3.2.7 : Find f' for the function $f(x)=2x(3x^5+\frac{3}{x})$

Solution:

$$f'(x)= 2x(15x^4-\frac{3}{x^2})+(3x^5+\frac{3}{x}).2$$

$$= 30x^5-\frac{6}{x}+6x^5+\frac{6}{x} = 36x^5$$

Example 3.2.8 : Find f' for the function $f(x)=\frac{2x-1}{3x+1}$

Solution:

$$f'(x) = \frac{(3x+1).2 - (2x-1).3}{(3x+1)^2}$$

$$= \frac{6x+2-6x+3}{(3x+1)^2} = \frac{5}{(3x+1)^2}$$

The derivative of the cosine function is the negative of the sine function :

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 3.2.10 : Find $f'(x)$ for the function $f(x) = 3x^2 + 2\cos x$

Solution: $f'(x) = 6x - 2\sin x$

Example 3.2.11 : Find y' for each of the following functions :

(i) $y = \sin x - \cos x$ *(ii)* $y = 2\sin x \cos x$ *(iii)* $y = \dfrac{3\sin x}{\cos x + 1}$

Solution:

(i) $y' = \cos x + \sin x$

(ii) $y' = 2\sin x \,.\, (-\sin x) + \cos x \,.\, (2\cos x) = -2\sin^2 x + 2\cos^2 x$

(iii) $y' = \dfrac{(\cos x + 1).(3\cos x) - (3\sin x).(-\sin x)}{(\cos x + 1)^2}$

$= \dfrac{3\cos^2 x + 3\cos x + 3\sin^2 x}{(\cos x + 1)^2}$

The derivative of other trigonometric functions :

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 3.2.12 : Find y' for each of the following functions :

(i) $y = \tan x + \sec x$ *(ii)* $y = 5\cot x \csc x$

Solution:

$(i)\ y' = \sec^2 x + \sec x \tan x$

$(ii)\ y' = 5\cot x\,.\,(-\csc x \cot x) + \csc x\,.\,(-5\csc^2 x)$
$= -5\csc x \cot^2 x - 5\csc^3 x$

Derivative of Logarithmic Function:

The derivative of the natural logarithmic function is:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Example 3.2.13 : Find y' for each of the following functions :

$(i)\ y = 4x^3 \ln x \qquad (ii)\ y = \frac{2\ln x}{9x+1}$

Solution:

(i) $y' = 4x^3(\frac{1}{x}) + \ln x(12x^2) = 4x^2 + 12x^2 \ln x$

$(ii)\ y' = \frac{(9x+1)(\frac{2}{x}) - (2\ln x)(9)}{(9x+1)^2} = \frac{18 + \frac{2}{x} - 18\ln x}{(9x+1)^2}$

Derivative of Exponential Function :

The derivative of the exponential functions are:

$$\frac{d}{dx}a^x = a^x \ln a \quad \text{and} \quad \frac{d}{dx}e^x = e^x$$

Example 3.2.14 : Find f' for the function $f(x) = 5x^7 e^x + 4e^x$.

Solution: $f'(x) = 5x^7 e^x + e^x(35x^6) + 4e^x = 5x^7 e^x + 35x^6 e^x + 4e^x$

Implicit Differentiation (Derivative of Composite Functions) :

Chain Rule :

Let $y=f(u)$, $u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Example 3.2.15 : Let $y=6u^3+5u$, $u=\ln x$, find $\frac{dy}{dx}$.

Solution:

$\frac{dy}{du}=18u^2+5$, $\frac{du}{dx}=\frac{1}{x}$

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=(18u^2+5)(\frac{1}{x})=\left(18(\ln x)^2+5\right)(\frac{1}{x})$$

$$=\frac{18}{x}(\ln x)^2+\frac{5}{x}$$

Example 3.2.16 : Find $\frac{dy}{dx}$ for each of the following functions :

(i) $y=(x+4x^3)^6$, (ii) $y=\ln(x^2+3)$, (iii) $y=\tan^3 x$.

Solution:

(i) let $u=x+4x^3$, then $y=u^6$.

Thus $\frac{dy}{du}=6u^5$ and $\frac{du}{dx}=1+12x^2$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=6u^5(1+12x^2)=6(x+4x^3)^5(1+12x^2) .$$

(ii) let $u=x^2+3$, then $y=\ln u$.

Thus $\frac{dy}{du}=\frac{1}{u}$ and $\frac{du}{dx}=2x$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=\frac{1}{u}(2x)=\frac{2x}{x^2+3}$$

(iii) let $u=\tan x$, then $y=u^3$.

Thus $\frac{dy}{du}=3u^2$ and $\frac{du}{dx}=\sec^2 x$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=3u^2(\sec^2 x)=3\tan^2 x\sec^2 x$$

In examples (3.2.15 and 3.2.16) we use the Chain rule to get the derivative of a composite function using substitutions, but also we can get the same results directly without substitutions, considering the following rules:

$$\frac{d}{dx}(f(x))^n = n\,(f(x))^{n-1}\,.\,f'(x),$$

$$\frac{d}{dx}(\,\ln f(x)\,) = \frac{1}{f(x)}\,.\,f'(x),$$

$$\frac{d}{dx}(\,e^{f(x)}\,) = e^{f(x)}\,.\,f'(x),$$

$$\frac{d}{dx}(\,\sin f(x)\,) = (\,\cos f(x)).\,f'(x),$$

$$\frac{d}{dx}(\,\cos f(x)\,) = (\,-\sin f(x))\,.\,f'(x),$$

$$\frac{d}{dx}(\,\tan f(x)\,) = (\,\sec^2 f(x))\,.\,f'(x),$$

$$\frac{d}{dx}(\,\sec f(x)\,) = (\,\sec f(x)\,.\,\tan f(x))\,.\,f'(x),$$

$$\frac{d}{dx}(\,\csc f(x)\,) = (\,-\csc f(x)\,.\,\cot f(x))\,.\,f'(x),$$

$$\frac{d}{dx}(\,\cot f(x)\,) = (\,-\csc^2 f(x))\,.\,f'(x).$$

Example 3.2.17 : Find $\frac{dy}{dx}$ for each of the following functions :

$(i)\ y = \sqrt{x^5 + 4x}$, $(ii)\ y = \ln(\,x^2 + 3x\,)$, $(iii)\ y = e^{3x}$.

Solution:

$(i)\ y = \sqrt{x^5 + 4\,x} = (\,x^5 + 4\,x\,)^{\frac{1}{2}}$

$$\therefore\ \frac{dy}{dx} = \frac{1}{2}\,(\,x^5 + 4\,x\,)^{-\frac{1}{2}}.\,(\,5\,x^4 + 4\,) = \frac{5x^4 + 4}{2\sqrt{x^5 + 4x}}\ .$$