CHAPTER ONE

The Errors

Definition 1.1: An algorithm is a procedure or formula for solving a problem.

Definition 1.2: Numerical analysis is a branch of mathematics concern with the study of algorithms that use numerical approximation solution for the problems.

Type of Errors

Definition 1.3: Absolute error is the absolute value of the difference between the true value x and the approximate value $\hat{x}$, and denoted by $e_a(x)$ (i.e. $e_a(x) = |\Delta x| = |x - \hat{x}|$)

Example 1.4 If $x = 100$ and $\hat{x} = 97$, find the absolute error of x.

Solution: The absolute error of x is

$e_a(x) = |\overset{x}{100} - \overset{\hat{x}}{97}| = 3$.

Definition 1.5: The relative error of x is the ratio of the absolute error $e_a(x)$

to the absolute value of the true value x (where $x \neq 0$) and denoted by $e_r(x)$ or δ_x (i.e. $\delta_x = e_r(x) = \frac{e_a(x)}{|x|} = \frac{|x-\hat{x}|}{|x|} = \left|\frac{x-\hat{x}}{x}\right|$, where $x \neq 0$).

Example 1.6: If $x = 100$ and $\hat{x} = 97$, find the relative error of x.

Solution:

The relative error of x is

$$e_r(x) = \delta_x = \left|\frac{\overset{x}{100} - \overset{\hat{x}}{97}}{\underset{x}{100}}\right| = \left|\frac{3}{100}\right| = 0.03.$$

Example 1.7: If $x = 0.008$ and $\hat{x} = 0.007$, find the absolute error of x and the relative error of x.

Solution: The absolute error of x is

$$e_a(x) = |\overset{x}{0.008} - \overset{\hat{x}}{0.007}| = 0.001,$$

and the relative error of x is

$$\delta_x = e_r(x) = \left|\frac{x-\hat{x}}{x}\right| = \left|\frac{0.008-0.007}{0.008}\right|$$
$$= \left|\frac{0.001}{0.008}\right| = \frac{1}{8} = 0.125$$

Definition 1.8: The percentage error of x is the relative error multiplied by 100, which is denoted by $e\%$, i.e. $e\% = e_r(x) * 100$

$$= \left|\frac{x-\hat{x}}{x}\right| * 100.$$

Sources of Errors:

(1) Round-off Error

(2) Truncation Error

Definition 1.8: A computer or P.C. represent any number approximately. The true value minus the approximate value is called the round-off error.

Example 1.9: Find the round-off error of $\frac{1}{3}$.

Solution: The round off error of $\frac{1}{3}$ is

$$\frac{1}{3} - 0.3333333 = 0.0000000333\cdots$$

Remark 1.10:

1) If the approximate value of $\sqrt{2}$ is 1.41421 then the round-off error of $\sqrt{2}$ is

$$\sqrt{2} - 1.41421 = 0.000003562\cdots$$

2) If the approximate value of π is 3.14159 then the round-off error of π is

$$\pi - 3.14159 = 0.000002654\cdots$$

Definition 1.11: If we take a finite summation to approximate the summation of an infinite series then the summation of the infinite series minus the approximation summation (which is the remainder of the infinite series) is called the **truncation error**.

Remarks

1) The Taylor series generated by f at $x=a$ is

$$f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

2) The Maclaurin series generated by f is

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

Example 1.12: Find the truncation error of $e^{0.5}$ when the approximation of $e^{0.5}$ is the summation of the first three terms of the Maclaurin series of e^x.

Solution: The Maclaurin series of e^x is

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

Then $e^{0.5} \approx 1+0.5+\frac{(0.5)^2}{2} = 1+0.5+0.125$
$= 1.625$

While $e^{0.5} = 1.648721271$

Then the truncation error is

$1.648721271 - 1.625 = 0.023721271$

$$= \frac{(0.5)^3}{3!}+\frac{(0.5)^4}{4!}+\cdots$$