FUNCTIONAL ANALYSIS
With Applications

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Hilbert Space

In this chapter, we study a special Banach space which possesses an additional structure known as the *inner product*. This enables us to generalize several geometric concepts—in particular, the concepts of length and angle of Euclidean spaces to infinite-dimensional spaces. The well-known parallelogram law and several other geometric relations of the plane are proved for this special class of Banach spaces which is known as the Hilbert space. The concept of orthogonality in such spaces leads to the celebrated 'projection theorem' and theory of Fourier series generalizing several classical results concerning the trigonometric Fourier series. The additional structure has important implications, such as every real Hilbert space is isometrically isomorphic to its dual and an arbitrary Hilbert space is reflexive. Besides these results, the Hilbert space exhibits some interesting properties of linear operators which are of vital importance for the study of certain systems occurring in physics and engineering. In the first five sections, we shall discuss the results mentioned above while the last section will be devoted to the study of bilinear forms and the Lax-Milgram lemma which are topics of current interest.

2.1 BASIC DEFINITION AND PROPERTIES

2.1.1 Definitions, Examples and Properties of Inner-product Space

Definition 2.1 Let X be a vector space over the field of real or complex numbers. A mapping, denoted by $\langle . , . \rangle$, defined on $X \times X$ into the underlying field is called the *inner product* of any two elements x and y of X if the following conditions are satisfied.

1. $\langle x+x', y\rangle = \langle x, y\rangle + \langle x', y\rangle$
2. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, α belongs to the underlying field
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$
4. $\langle x, x\rangle \geqslant 0$, $\forall\ x \in X$; and $\langle x, x\rangle = 0$ iff $x = 0$.

If the inner product $\langle .\,, . \rangle$ is defined for every pair of elements $(x, y) \in X \times X$, then the vector space X together with the inner product $\langle .\,, . \rangle$ is called an *inner-product space* or *pre-Hilbert space* usually denoted by $(X, \langle .\,, . \rangle)$.

Remark 2.1 1. $\langle x, y \rangle$ denotes the inner product of two vectors x and $y \in X$. Some authors use the symbol $\langle . / . \rangle$ or $(.\,, .)$ for the inner product.

2. $\overline{\langle x, y \rangle}$ denotes conjugate of the number $\langle x, y \rangle$. Therefore, if it is real, then $\overline{\langle x, y \rangle} = \langle x, y \rangle = \langle y, x \rangle$.

3. In view of condition (3), $\langle x, x \rangle$ must be real. (For $x = y$ in (3), we have $\langle x, x \rangle = \overline{\langle x, x \rangle}$. We know that a complex number z is real iff $z = \bar{z}$. Therefore, the number $\langle x, x \rangle$ must be real.)

4. The conditions (1) and (2) imply that the function $\langle .\,, . \rangle$ is linear in the first variable x. It is easy to see that $\langle .\,, . \rangle$ is also linear in the second variable y if X is a real vector space. In fact.

(a) $\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle \ \forall\ x, y, y' \in X$

(b) $\langle x, \beta y \rangle = \bar{\beta} \langle x, y \rangle \ \forall\ x, y \in X$ and β belonging to the underlying field.

Verification of (a). $\langle x, y + y' \rangle = \overline{\langle y + y', x \rangle}$ by condition (3) of Definition 2.1. By condition (1), $\langle y + y', x \rangle = \langle y, x \rangle + \langle y', x \rangle$, and so $\langle x, y + y' \rangle = \overline{\langle y, x \rangle + \langle y', x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle y', x \rangle}$ by the property of the conjugate of the complex numbers. Again by condition (3), $\overline{\langle y, x \rangle} = \langle x, y \rangle$ and $\overline{\langle y', x \rangle} = \langle x, y' \rangle$.

Therefore, $$\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle.$$

Verification of (b). $\langle x, \beta y \rangle = \overline{\langle \beta y, x \rangle}$, by condition (3) but $\langle \beta y, x \rangle = \beta \langle y, x \rangle$ by condition (2). Therefore, $\langle x, \beta y \rangle = \overline{\beta \langle y, x \rangle} = \bar{\beta}\, \overline{\langle y, x \rangle}$, $= \bar{\beta} \langle x, y \rangle$, using the properties of conjugate numbers and condition (3). (For properties of conjugate numbers, see Appendix D.)

5. Using the first and second conditions of the inner product, (a) and (b) of the preceding remark, it can be seen that:

$$\left\langle \sum_{k=1}^{n} \alpha_k x_k, \sum_{l=1}^{n} \beta_l y_l \right\rangle = \sum_{k=1}^{n} \sum_{l=1}^{n} \alpha_k \bar{\beta}_l \langle x_k, y_l \rangle$$

6. $\langle x, 0 \rangle = 0, \ \forall\ x \in X$.

Verification of (6). $\langle x, 0 \rangle = \overline{\langle 0, x \rangle} = \overline{\langle 0 + 0, x \rangle} = \overline{\langle 0, x \rangle} + \overline{\langle 0, x \rangle} = \langle x, 0 \rangle + \langle x, 0 \rangle = 2 \langle x, 0 \rangle$. If $\langle x, 0 \rangle \neq 0$, then $1 = 2$ (an absurdity). So $\langle x, 0 \rangle = 0 \ \forall\ x \in X$.

7. If, for a given element $y \in X$, $\langle x, y \rangle = 0 \ \forall\ x \in X$, then $y = 0$. ($\langle x, y \rangle = 0$ is valid for $x = y$, i.e., $\langle y, y \rangle = 0$, which implies that $y = 0$.)

8. The inner product is a continuous function with respect to the norm induced by it. (For the proof, see Example 2.25.)

EXAMPLE 2.1 R^2 is an inner product space with the inner product.

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2$$

where $x = (x_1, x_2) \in R^2$
$y = (y_1, y_2) \in R^2$

EXAMPLE 2.2 R^n, $n \geqslant 1$ is an inner product space with the inner product.

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$
$y = (y_1, y_2, \ldots, y_n) \in R^n$

EXAMPLE 2.3 The vector space $C[a, b]$ of continuous functions defined on $[a, b]$ is an inner-product space with the inner product:

1. $\langle f, g \rangle = \int_a^b f(x)\, \overline{g(x)}\, dx$, where $f, g \in C[a, b]$.
2. $\langle f, g \rangle = \int_a^b f(t)\, \overline{g(t)})w(t)\, dt$, where $f, g \in C[a, b]$ and $w(t) \geqslant 0$ and belongs to $C[a, b]$.

For $w(t) = 1$, we get the inner product of (1). $w(t)$ is called a *weight function.*

EXAMPLE 2.4 $l_2 = \left\{ x = \{x_n\} \middle/ \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ is an inner product space with the inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$.

where $x = \{x_i\} \in l_2$
$y = \{y_i\} \in l_2$

We call l_2 real if the sequences $\{x_n\}$ are real.

EXAMPLE 2.5 $L_2 = \{$All Lebesgue integrable functions (real or complex value) over $(a, b)/|f|^2$ is also Lebesgue integrable over $(a, b)\}$ is an inner-product space with the inner product $\langle f, g \rangle = \int_a^b f\bar{g}\, dx$ where $f, g \in L_2(a, b)$.

NOTE For the details of spaces $C[a, b]$, l_2, $L_2(a, b)$ one may see Appendix D. The verifications of these examples are given in the solved examples of this chapter. It will be seen in Sec. 2.1.3 that L_p and l_p, $p \neq 2$, are not inner-product spaces.

Remark 2.2 1. An inner-product space will be called *finite dimensional* if the underlying vector space is *finite dimensional.* Sometimes a finite-dimensional inner-product space over the field of complex numbers is called a *Hermitian space* or *unitary space.* A finite-dimensional inner-product space over the field of real numbers is called a *Euclidean space.*

2. Unless explicitly mentioned all abstract Hilbert spaces are defined over the field of complex numbers.

Theorem 2.1 (Cauchy-Schwartz-Bunyakowski inequality). For all x, y belonging to an inner-product space X, we have

$$|\langle x, y\rangle|^2 \leqslant \langle x, x\rangle \langle y, y\rangle \tag{2.1}$$

Proof If $y=0$, then $\langle x, y\rangle=0$, and $\langle y, y\rangle=0$ by Remark 2.1(6). Therefore, Eq. 2.1 is satisfied as both sides are zero.

Let $y \neq 0$, then $\langle y, y\rangle \neq 0$. Let $\lambda=\dfrac{\langle x, y\rangle}{\langle y, y\rangle}$. Then we have

$$\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}=\frac{\langle x, y\rangle \overline{\langle x, y\rangle}}{\langle y, y\rangle}$$

$$=\lambda \overline{\langle x, y\rangle}=\lambda \langle y, x\rangle, \text{ by condition (4) of}$$

Definition 2.1

But we can write $\lambda \langle y, x\rangle=\bar{\lambda} \langle x, y\rangle$ by the same condition.

Thus,

$$\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}=\lambda \langle y, x\rangle=\bar{\lambda} \langle x, y\rangle=|\lambda|^2 \langle y, y\rangle \tag{2.2}$$

By using conditions (1) and (2) of Definition 2.1 and Remark 2.1 (4) (a) and (b), we have

$$0 \leqslant \langle x-\lambda y, x-\lambda y\rangle=\langle x, x\rangle+\langle x, -\lambda y\rangle+\langle -\lambda y, x\rangle+\langle -\lambda y, -\lambda y\rangle$$

$$=\langle x, x\rangle-\bar{\lambda} \langle x, y\rangle-\lambda \langle y, x\rangle+|\lambda|^2 \langle y, y\rangle \tag{2.3}$$

By Eqs 2.2 and 2.3, we obtain

$$0 \leqslant \langle x, x\rangle-\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}-\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}+\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}$$

or $$0 \leqslant \langle x, x\rangle-\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}$$

or $$|\langle x, y\rangle|^2 \leqslant \langle x, x\rangle \langle y, y\rangle \tag{2.4}$$

This is the desired inequality.

Theorem 2.2 Every inner-product space X is a normed space with respect to the norm $\|x\|=|\langle x, x\rangle|^{1/2}$ $\forall$ $x \in X$.

Proof Since the inner-product space X is a vector space, by definition, it is only required to verify axioms of the norm (Definition 1.1).

1. $\|x\| \geqslant 0$ $\forall$ x and $\|x\|=0$ iff $x=0$. $\|x\|=[\langle x, x\rangle]^{1/2}$, $\langle x, x\rangle \geqslant 0$, and $\langle x, x\rangle=0$ iff $x=0$ by condition (4) of the definition of the inner product. Therefore, (1) is satisfied.

2. $\|\alpha x\|=|\alpha|\, \|x\|$ $\forall$ $x \in X$ and α real or complex. We have $\|\alpha x\|=[\langle \alpha x, \alpha x\rangle]^{1/2}=[\alpha\bar{\alpha} \langle x, x\rangle]^{1/2}$ by condition (2) of Definition 2.1 and

Remark 2.1 (4) (b). Since $\alpha\bar{\alpha}=|\alpha|^2$, we have $\|\alpha x\|=[|\alpha|^2\langle x, x\rangle]^{1/2}=|\alpha|[\langle x, x\rangle]^{1/2}=|\alpha|\,\|x\|$.

3. $\|x+y\|\leq\|x\|+\|y\|\ \forall\ x, y\in X$. We have,

$$\begin{aligned}\|x+y\|^2&=\langle x+y, x+y\rangle\\&=\langle x, x\rangle+\langle x, y\rangle+\overline{\langle x, y\rangle}+\langle y, y\rangle\\&=\langle x, x\rangle+2\operatorname{Re}\langle x, y\rangle+\langle y, y\rangle\end{aligned}$$

by Definition 2.1, and Appendix D. By the Cauchy-Schwartz-Bunyakowski inequality, $\operatorname{Re}\langle x, y\rangle\leq|\langle x, y\rangle|\leq(\langle x, x\rangle)^{1/2}(\langle y, y\rangle)^{1/2}$. Therefore, we have

$$\begin{aligned}\|x+y\|^2&\leq\langle x, x\rangle+2(\langle x, x\rangle)^{1/2}(\langle y, y\rangle)^{1/2}+\langle y, y\rangle\\&=[[\langle x, x\rangle]^{1/2}+[\langle y, y\rangle]^{1/2}]^2\end{aligned}$$

or $$\|x+y\|\leq[\langle x, x\rangle]^{1/2}+[\langle y, y\rangle]^{1/2}=\|x\|+\|y\|$$

$\therefore$ $$\|x+y\|\leq\|x\|+\|y\|$$

This proves that $\|x\|=[\langle x, x\rangle]^{1/2}$ is a norm on X and $(X, \|\cdot\|)$ is a normed space.

Remark 2.3 1. In view of Th. 2.2, Cauchy-Schwartz-Bunyakowski inequality is written as $|\langle x, y\rangle|\leq\|x\|\,\|y\|$.

2. In the Cauchy-Schwartz-Bunyakowski inequality, equality holds, i.e. $|\langle x, y\rangle|=\|x\|\,\|y\|$, iff x and y are linearly dependent.

Verification Suppose $|\langle x, y\rangle|=\|x\|\,\|y\|$ and $x\neq 0$, $y\neq 0$. (If either $x=0$ or $y=0$ then x and y are linearly dependent and the desired result follows.)

This implies that $\langle x, y\rangle\neq 0$ and $\langle y, y\rangle\neq 0$. If $\lambda=\dfrac{\langle x, y\rangle}{\langle y, y\rangle}$, then $\lambda\neq 0$ and

$$\begin{aligned}\langle x-\lambda y, x-\lambda y\rangle&=\langle x, y\rangle-\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}\quad\text{[see (Eqs 2.3 and (2.4)]}\\&=\|x\|^2-\frac{\|x\|^2\,\|y\|^2}{\|y\|^2}=0\end{aligned}$$

This implies by Definition (2.1) (4) that $x-\lambda y=0$.

Hence, x and y are linearly dependent.

Conversely, if x and y are linearly dependent, we can write $x=\lambda y$. This implies that

$$\begin{aligned}|\langle x, y\rangle|&=|\langle\lambda y, y\rangle|\\&=|\lambda|\,|\langle y, y\rangle|\qquad\text{(by Definition 2.1 (2))}\\&=|\lambda|\,\|y\|^2=(|\lambda|\,\|y\|)\,\|y\|\qquad\text{by Th. 2.2}\\&=\|\lambda y\|\,\|y\|\\&=\|x\|\,\|y\|\end{aligned}$$

Remark 2.4 1. The norm $\| x \| = | \langle x, x \rangle |^{1/2}$ is said to be the norm induced by an inner product.

2. Since every inner-product space is a normed space, all the definitions given in Ch. 1 can be translated in an inner-product space with respect to the norm induced by an inner product. For example, a sequence $\{x_n\}$ in an inner-product space X is called a Cauchy sequence if for $\epsilon > 0$, there exists N, such that $\| x_n - x_m \| = [\langle x_n - x_m, x_n - x_m \rangle]^{1/2} < \epsilon$ for n and $m > N$, i.e. $\lim_{\substack{n\to\infty \\ m\to\infty}} \langle x_n - x_m, x_n - x_m \rangle = 0$.

2.1.2 Hilbert Space

Definition 2.2 An inner-product space X is called a *Hilbert space* if the normed space induced by the inner product is a Banach space (complete normed space). That is every Cauchy sequence $\{x_n\} \in X$ with respect to the norm induced by the inner product is convergent with respect to this norm.

EXAMPLE 2.6 1. R^n, l_2 and $L_2(a \cdot b)$ are Hilbert spaces.

2. $C[a, b]$ and $P[a, b]$ are inner-product spaces but not Hilbert spaces.

NOTE For $P[0, 1]$, see Appendix D (Definition 9) we define an inner product on $P[0, 1]$ as follows.

$$\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$$

where $f, g \in P[0, 1]$.

EXAMPLE 2.7 Let $Y = \left\{ f / f \text{ is absolutely continuous on } (a, b) \text{ with } f \text{ and } \frac{df}{dt} \text{ belonging to } L_2(a, b) \text{ and } f(a) = 0 = f(b) \right\}$

Y is a dense subspace of $L_2(a, b)$.

Y is a Hilbert space with respect to the following inner product.

$$\langle f, g \rangle_Y = \langle f, g \rangle + \left\langle \frac{df}{dt}, \frac{dg}{dt} \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L_2(a, b)$.

EXAMPLE 2.8 Let Ω be an open subset of R^3 and $C_0^\infty(\Omega)$, be the infinitely differentiable complex valued functions with compact support in Ω. Let

$$\langle f, g \rangle = \int_\Omega \left(f\bar{g} + \frac{\partial f}{\partial x_1} \frac{\partial \bar{g}}{\partial x_1} + \frac{\partial f}{\partial x_2} \frac{\partial \bar{g}}{\partial x_2} + \frac{\partial f}{\partial x_3} \frac{\partial \bar{g}}{\partial x_3} \right) dx_1\, dx_2\, dx_3$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $C_0^\infty(\Omega)$. This inner product induces the norm $\| f \| = \left(\int_\Omega (|f|^2 + |\nabla f|^2)\, dx_1\, dx_2\, dx_3 \right)^{1/2}$. However, $C_0^\infty(\Omega)$ is not a Hilbert space.

Remark 2.5 1. Historically l_2 is the first example of Hilbert space which was discovered by the celebrated German mathematician David Hilbert around 1910. Although the abstract axiomatization (Definition 2.1, 2.2) was given by J. von Neumann in 1927, the abstract space is itself called Hilbert space in view of his fundamental contribution concerning operators on such spaces.

2. In the early stages of the development of the Hilbert space theory, our present day Hilbert space (Definition 2.2) was assumed to satisfy an additional condition, viz. that the inner-product space is also separable. That is, up to 1930, a Hilbert space was a separable complete inner-product space. (Translate Definition 1.22 of a separable normed space for an normed space induced by a inner product.) In Example 2.27, we give the example of a Hilbert space that is not separable.

3. In Example 2.28, we shall show that every separable Hilbert space is isometric isomorphic to l_2.

2.1.3 Parallelogram Law and Characterization of Hilbert Space

From the elementary geometry (geometry of the plane), we know that the sums of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides. The following result gives a generalization of this result to an inner-product space.

Theorem 2.3 (Parallelogram law) For any two elements x and y belonging to an inner-product space, we have

$$\| x+y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2.$$

Proof Since X is an inner-product space, we have $\| x+y \|^2 = \langle x+y, x+y \rangle$ (Th. 2.2, and $\| x-y \|^2 = \langle x-y, x-y \rangle$.
By Definition 2.1 and Remark 2.1 (4), we have

$$\| x+y \|^2 = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

or

$$\| x+y \|^2 = \| x \|^2 + \langle x, y \rangle + \langle y, x \rangle + \| y \|^2 \tag{2.5}$$

Similarly,

$$\| x-y \|^2 = \| x \|^2 - \langle x, y \rangle - \langle y, x \rangle + \| y \|^2 \tag{2.6}$$

By adding Eqs 2.5 and 2.6, we obtain

$$\| x+y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2.$$

This is the desired result.

Remark 2.6 The following example shows that the parallelogram law is not valid for an arbitrary norm on a vector space.

Let $X = C[0, 2\pi]$ = space of all real-valued continuous function on $[0, 2\pi]$. Then, $C[0, 2\pi]$ is a Banach space with the norm $\| f \| = \sup_{0 \leq t \leq 2\pi} | f(t) |$. However this norm does not satisfy the parallelogram law. For example, choose $f(t)=\max(\sin t, 0)$ and $g(t)=\max(-\sin t, 0)$. Then

$$\|f\|=1,\ \|g\|=1,\ \|f+g\|=1, \text{ and } \|f-g\|=1.$$

Thus, $\quad 2\|f\|^2+2\|g\|^2=4, \quad \text{and} \quad \|f+g\|^2+\|f-g\|^2=2$

Hence, $\quad \|f+g\|^2+\|f-g\|^2 \neq 2\|f\|^2+2\|g\|^2.$

In fact, we shall prove in Th. 2.5 that if the norm of a normed space satisfies the parallelogram identity, the space is an inner-product space.

Theorem 2.4 (Polarization identity) For any two elements x and y belonging to an inner-product space, we have

$$\langle x, y\rangle=\frac{1}{4}\{\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2\}.$$

Proof By Definition 2.1 and Remark 2.1 we have

$$\|x+y\|^2=\langle x, x\rangle+\langle x, y\rangle+\langle y, x\rangle+\langle y, y\rangle$$
$$-\|x-y\|^2=-\langle x, x\rangle+\langle x, y\rangle+\langle y, x\rangle-\langle y, y\rangle$$
$$i\|x+iy\|^2=i\langle x, x\rangle+\langle x, y\rangle-\langle y, x\rangle+i\langle y, y\rangle$$
$$-i\|x-iy\|^2=-i\langle x, x\rangle+\langle x, y\rangle-\langle y, x\rangle-i\langle y, y\rangle$$

By adding these four relations, we obtain

$$\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2=4\langle x, y\rangle$$

or $$\langle x, y\rangle=\frac{1}{4}[\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2]$$

Theorem 2.5 [Jordan-von Neumann, 1935]. A normed space is an inner-product space iff the norm of the normed space satisfies the parallelogram law.

Theorem 2.6 A Banach space is a Hilbert space iff its norm satisfies the parallelogram law.

Proof of Th. 2.5 1. Let X be an inner-product space. Then, by Th. 2.3, the parallelogram law is satisfied.

2. Conversely, suppose that X is a normed space such that its norm $\|\cdot\|$ satisfies the parallelogram law. That is, we have the relation

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \quad \forall\ x, y\in X.$$

Now, we want to show that an inner product is defined on X.

Let $$\varphi(x, y)=\frac{1}{4}\{\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2\}.$$

Now, we prove that $\varphi(x, y)$ is an inner product on X, i.e. it satisfies the following conditions.

(a) $\varphi(x+x', y)=\varphi(x, y)+\varphi(x', y)$

(b) $\varphi(\alpha x, y)=\alpha\varphi(x, y)$

(c) $\varphi(x, y) = \overline{\varphi(y, x)}$

(d) $\varphi(x, x) \geqslant 0$, and $\varphi(x, x) = 0$ iff $x = 0$

Verification of (a) (a) is equivalent to

$$\frac{1}{4}[\| x+x'+y \|^2 - \| x+x'-y \|^2 + i \| x+x'+iy \|^2 - i \| x+x'-iy \|^2]$$

$$= \frac{1}{4}[\| x+y \|^2 - \| x-y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2]$$

$$+ \frac{1}{4}[\| x'+y \|^2 - \| x'-y \|^2 + i \| x'+iy \|^2 - i \| x'-iy \|^2] \quad (2.7)$$

In order to verify Eq. 2.7, it is sufficient to verify that the real and imaginary parts of its right-hand side are respectively equal to the real and imaginary parts of its left-hand. In other words, the verification of Eq. 2.7 is equivalent to the verification of the following relations.

$$\| x+x'+y \|^2 - \| x+x'-y \|^2 = \| x+y \|^2 - \| x-y \|^2 + \| x'+y \|^2 - \| x'-y \|^2 \quad (2.8)$$

and $$\| x+x'+iy \|^2 - \| x+x'-iy \|^2 = \| x+iy \|^2 - \| x-iy \|^2 + \| x'+iy \|^2 - \| x'-iy \|^2 \quad (2.9)$$

Verification of Eq. 2.8 By the parallelogram law for the elements $\frac{x'}{2}+x$ and $\frac{x'}{2}+y$, we have

$$\| x+x'+y \|^2 + \| y-x \|^2 = 2 \left\| \left(x+\frac{x'}{2}\right) \right\|^2 + 2 \left\| \left(y+\frac{x'}{2}\right) \right\|^2.$$

Since $\| y-x \| = \| x-y \|$, we have

$$\| x+x'+y \|^2 + \| x-y \|^2 = 2 \left\| x+\frac{x'}{2} \right\|^2 + 2 \left\| y+\frac{x'}{2} \right\|^2 \quad (2.10)$$

Replacing y by $-y$ in Eq. 2.10, we obtain

$$\| x+x'-y \|^2 + \| x+y \|^2 = 2 \left\| x+\frac{x'}{2} \right\|^2 + 2 \left\| \frac{x'}{2}-y \right\|^2 \quad (2.11)$$

Equation 2.8 takes the following form if we put the values of $\| x+x'+y \|^2$ and $\| x+x'-y \|^2$ given by Eqs 2.10 and 2.11 respectively.

$$2 \left\| x+\frac{x'}{2} \right\|^2 + 2 \left\| y+\frac{x'}{2} \right\|^2 - \| x-y \|^2 + \| x+y \|^2 - 2 \left\| x+\frac{x'}{2} \right\|^2$$

$$-2 \left\| \frac{x'}{2}-y \right\|^2 = \| x+y \|^2 - \| x-y \|^2 + \| x'+y \|^2 - \| x'-y \|^2$$

or $$2 \left\| y+\frac{x'}{2} \right\|^2 - 2 \left\| \frac{x'}{2}-y \right\|^2 = \| x'+y \|^2 - \| x'-y \|^2 \quad (2.12)$$

By the parallelogram law for the elements $\frac{x'}{2}+y$ and $\frac{x'}{2}$, we have

$$\| x'+y \|^2+\| y \|^2=2 \| \frac{x'}{2}+y \|^2+2 \| \frac{x'}{2} \|^2 \tag{2.13}$$

By changing y to $-y$ in Eq. 2.13, we obtain

$$\| x'-y \|^2+\| y \|^2=2 \| \frac{x'}{2}-y \|^2+2 \| \frac{x'}{2} \|^2 \qquad (\text{Recall } \| -y \| = \| y \|) \tag{2.14}$$

Subtracting Eq. 2.14 from Eq. 2.13, we get

$$\| x'+y \|^2-\| x'-y \|^2=2 \| \frac{x'}{2}+y \|^2-2 \| \frac{x'}{2}-y \|^2 \tag{2.15}$$

Equation 2.15 is equivalent to Eq. 2.12 which in turn is equivalent to Eq. 2.8. Hence, Eq. 2.8 is verified.
Replacing y by iy in Eq. 2.8, we get 2.9. Thus,

$$\varphi(x+x', y)=\varphi(x, y)+\varphi(x', y)$$

Verification of (b) We verify the relation for different values of α.
Case (i): If $\alpha=0$, then $\varphi(\alpha x, y)=\varphi(0, y)=0$ and $\alpha\varphi(x, y)=0$. Hence, the desired relation is true.
Case (ii): $\alpha=-1$. By (a) we have $\varphi(x+x', y)=\varphi(x, y)+\varphi(x', y)$. If $x'=-x$ in this relation, $\varphi(0, y)=\varphi(x, y)+\varphi((-1)x, y)$ or $(-1)\varphi(x, y)=\phi((-1)x, y)$.
Case (iii) $\alpha=p$, where p is a natural number. Then, by the principle of induction, (a) gives $\varphi(px, y)=p\varphi(x, y)$.
Case (iv): $\alpha=n$, where n is integer, i.e. $\alpha=\pm p$.
Cases (iii) and (iv) imply that

$$\varphi(nx, y)=n\varphi(x, y)$$

Case (v): $\alpha=n/m$, i.e., α is a rational number.
We have $\varphi(n/mx, y)=n\phi\left(\frac{x}{m}, y\right)=\frac{n}{m}m\varphi\left(\frac{x}{m}, y\right)=\frac{n}{m}\varphi\left(m\frac{x}{m}, y\right)$,
by case (iv)

Hence, $$\varphi\left(\frac{n}{m}x, y\right)=\frac{n}{m}\varphi(x, y)$$

Case (vi): α is a real number. We know that every real number can be expressed as the limit of a sequence of rational numbers, i.e. $\alpha=\lim_{n\to\infty} \gamma_n$, γ_n's being rational numbers.

We are required to verify that $\varphi(\alpha x, y)=\alpha\varphi(x, y)$ or $\varphi(\lim_{n\to\infty} \gamma_n x, y)=\lim_{n\to\infty} \gamma_n\varphi(x, y)$. Since $\varphi(\gamma_n, y)=\gamma_n(x, y)$ for each n, it is sufficient to verify that $\varphi(\alpha x, y)=\lim_{n\to\infty} \varphi(\gamma_n x, y)$. In view of (a), it is sufficient to verify that $\lim_{n\to\infty} \varphi((\alpha-\gamma_n)x, y)=0$.

In order to verify it, we are required to show that $\varphi(\mu x, y) \to 0$ as $\mu \to 0$, which is equivalent to verify that $\lim_{\mu \to 0} \| \mu x + y \| = \| y \|$.

By the triangular inequality of the norm, we have

$$\big|\, |\mu| \| x \| - \| y \| \,\big| = \big|\, \| \mu x \| - \| y \| \,\big| \leqslant \| \mu x + y \| \leqslant |\mu| \| x \| + \| y \|$$

By taking the limit in this relation as $\mu \to 0$, we get $\| y \| \leqslant \lim_{\mu \to 0} \| \mu x + y \| \leqslant \| y \|$ or $\lim_{\mu \to 0} \| \mu x + y \| = \| y \|$.

Case (vii): $\alpha = i$

$$\varphi(ix, y) = \frac{1}{4}[\| ix + y \|^2 + \| ix - y \|^2 + i \| ix + iy \|^2 - i \| ix - iy \|^2]$$

$$= i \frac{1}{4}[\| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2]$$

$$= i\varphi(x, y)$$

Case (viii). $\alpha = n + im$.

$$\varphi((n+im)x, y) = \varphi(nx + imx, y)$$
$$= \varphi(nx, y) + \varphi(imx, y)$$
$$= n\varphi(x, y) + i\varphi(mx, y) \qquad \text{by Case (vii)}$$
$$= n\varphi(x, y) + im\varphi(x, y)$$
$$= (n+im)\varphi(x, y)$$
$$= \alpha\varphi(x, y)$$

Verification of (c) We want to verify that $\varphi(x, y) = \overline{\varphi(y, x)}$.
We have

$$\overline{\varphi(y, x)} = \text{Conjugate of } \frac{1}{4}[\| y + x \|^2 - \| y - x \|^2 + i \| y + ix \|^2 - i \| y - ix \|^2]$$

$$= \frac{1}{4}[\| x + y \|^2 - \| x - y \|^2 - i \| y + ix \|^2 + i \| y - ix \|^2]$$

$$= \frac{1}{4}[\| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x + iy \|^2] = \varphi(x, y)$$

As $\| y - x \| = \| x - y \|$,

$$-i \| y + ix \| = -i \| (-i^2) y + ix \| = (-i) \| i(x - iy) \|^2$$
$$= (-i)[\, | i |^2 \| x - iy \|^2] = -i \| x - iy \|$$

and

$$i \| y - ix \| = i \| (-1)i^2 y - ix \| = i \| (-i)(x + iy) \|^2$$
$$= i[\, | -i |^2 \| x + iy \|^2]$$
$$= i \| x + iy \|^2$$

Verification of (d)

$$\varphi(x, x) = \frac{1}{4}[\| 2x \|^2 + i \| x + ix \|^2 - i \| x - ix \|^2]$$

$$= \frac{1}{4}[4 \| x \|^2 + i \| (1+i)x \|^2 - i \| (1-i)x \|^2]$$

$$= \frac{1}{4}[4\|x\|^2 + i\,|1+i|^2\|x\|^2 - i\,|(1-i)|^2\|x^2\|^2]$$

$$= \frac{1}{4}[4\|x\|^2 + i\,2\|x\|^2 - i\,2\|x\|^2]$$

$$= \|x\|^2$$

or $\qquad \|x\| = [\varphi(x, x)]^{1/2}$

If $\varphi(x, x) = 0$, then $\|x\| = 0$. But $\|x\| = 0$ iff $x = 0$. Hence, $\varphi(x, x) = 0$ iff $x = 0$.

This proves the theorem.

Proof of Th. 2.6 1. Suppose X is a Hilbert space. Then by definition, it is a normed space. Also, by Th. 2.3, the parallelogram law is satisfied.

2. Suppose X is a Banach space such that its norm satisfies the parallelogram law. Since every Banach space is a normed space, by Th. 2.5,

$$\varphi(x, y) = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2] \qquad (2.16)$$

is an inner product on X.

Suppose $\{x_n\}$ is a Cauchy sequence in the inner-product space (X, φ), i.e.

$$\varphi(x_n - x_m, x_n - x_m) \to 0, \text{ as, } n, m \to \infty \qquad (2.17)$$

In view of Eqs 2.16 and 2.17, $\{x_n\}$ is convergent in (X, φ). Hence, (X, φ) is a Hilbert space.

Remark 2.7 1. l_p^n (see Example 1.4 and note after it) is not a Hilbert space for $p \neq 2$. For this, apply Th. 2.6.

Let $x = (1, 1, 0, 0, \ldots)$ and $y = (1, -1, 0, \ldots)$.

Then, $x + y = (2, 0, 0, \ldots)$ and $x - y = (0, 2, 0, \ldots)$.

We have

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} = (|1|^p + |1|^p + 0 + 0 + \ldots + 0)^{1/p}$$

$$= 2^{1/p}$$

$$\|y\| = (|1|^p + |1-1|^p + \ldots + 0) = 2^{1/p}$$

$$\|x+y\| = (|2|^p + 0 + \ldots + 0)^{1/p} = 2$$

$$\|x-y\| = (0 + |2|^p + \ldots + 0)^{1/p} = 2$$

Hence, $\|x+y\|^2 + \|x-y\|^2 = 8$ and $2\|x\|^2 + 2\|y\|^2 = 2(2^{2/p} + 2^{2/p})$.

If $p = 2$, the parallelogram law is satisfied, which implies that l_2^n is a Hilbert space. If $p \neq 2$, the parallelogram law is not satisfied. Therefore, in view of Th. 2.6, l_p^n, $p \neq 2$, is not a Hilbert space.

2. In view of Remark 2.6 and Th. 2.6, $C[a, b]$ with sup norm (Example (1.8) is not a Hilbert space.

2.2 ORTHOGONAL COMPLEMENTS AND PROJECTION THEOREM

2.2.1 Orthogonal Complements and Projections

Definition 2.3 1. Two vectors x and y in an inner-product space are called *orthogonal*, denoted by $x \perp y$, if $\langle x, y \rangle = 0$.

2. A vector x of an inner-product space X is called *orthogonal* to a nonempty subset A of X, denoted by $x \perp A$, if $\langle x, y \rangle = 0$ for each $y \in A$.

3. Let A be a nonempty subset of an inner-product space X. Then, the set of all vectors orthogonal to A, denoted by $A^\perp$, is called the *orthogonal complement* of A. It is clear that $A^\perp = \{x \in X / \langle x, y \rangle = 0 \text{ for each } y \in A\}$. $A^{\perp\perp} = (A^\perp)^\perp$ will denote orthogonal complement of $A^\perp$.

4. Two subsets A and B of an inner-product space X are called *orthogonal* denoted by $A \perp B$ if $\langle x, y \rangle = 0 \ \forall \ x \in A$ and $\forall \ y \in B$.

Remark 2.8 1. Since $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle x, y \rangle = 0$ implies that $\overline{\langle y, x \rangle} = 0$ or $\langle y, x \rangle = 0$ and vice versa. (see Appendix D 3(6).) Hence $x \perp y$ iff $y \perp x$.

2. In view of Remark 2.1(6), $x \perp 0$ for every x belonging to an inner-product space. By condition (4) of the definition of the inner product, 0 is the only vector orthogonal to itsself.

3. It is clear that $\{0\}^\perp = X$ and $X^\perp = \{0\}$.

4. It is clear that if $A \perp B$ then $A \cap B = \{0\}$.

5. Nonzero orthogonal vectors, $x_1, x_2, x_3, \ldots, x_n$ of an inner-product space are linearly independent.

Theorem 2.7 Let X be an inner-product space and A be its arbitrary subset. Then the following results hold good.

1. $A^\perp$ is a closed subspace of X.
2. $A \cap A^\perp \subset \{0\} \cdot A \cap A^\perp = \{0\}$ iff A is a subspace.
3. $A \subset A^{\perp\perp}$.
4. If $B \subset A$, then $B^\perp \supset A^\perp$.

Proof 1. Let $x, y \in A^\perp$. Then, $\langle x, z \rangle = 0 \ \forall z \in A$ and $\langle y, z \rangle = 0 \ \forall z \in A$. Since for arbitrary scalars α, β, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, by Definition 2.1, we get $\langle \alpha x + \beta y, z \rangle = 0$. i.e., $\alpha x + \beta y \in A^\perp$. So A is a subspace of X.

For showing that $A^\perp$ is closed, let $\{x_n\} \in A$ and $x_n \to y$.

We are required to show that y must belong to $A^\perp$.

By the definition of $A^\perp$, for every $x \in X \ \langle x, x_n \rangle = 0 \ \forall n$. This implies that $\lim_{n\to\infty} \langle x, x_n \rangle = \lim_{n\to\infty} \langle x_n, x \rangle = 0$ (Remark 2.8). Since $\langle \cdot, \cdot \rangle$ is a continuous function, $\langle \lim_{n\to\infty} x_n, x \rangle = 0$.

or $\langle y, x \rangle = 0$. Hence, $y \in A^\perp$.

2. If $y \in A \cap A^\perp$, i.e. $y \in A$ and $y \in A^\perp$; by Remark 2.8, $y = 0$, i.e., $y \in \{0\}$.

If A is a subspace, then $0 \in A$ and $0 \in A \cap A^\perp$. Hence, $A \cap A^\perp = \{0\}$.

3. Let $y \in A$, but $y \notin A^{\perp\perp}$. Then there exists an element $z \in A^\perp$ such that $\langle y, z \rangle \neq 0$. Since $z \in A^\perp$, $\langle z, y \rangle = 0$ which is a contradiction. Hence, $y \in A^{\perp\perp}$.

4. Let $y \in A^\perp$. Then $\langle y, z \rangle = 0 \; \forall z \in A$. Since every $z \in B$ is an element of A, we have $\langle y, z \rangle = 0 \; \forall z \in B$. Hence, $y \in B^\perp$, and so $B^\perp \supset A^\perp$.

This completes the proof of the theorem.

Definition 2.4 The angle θ between two vectors x and y of an inner-product space X is defined by the following relation

$$\cos \theta = \frac{\langle x, y \rangle}{\| x \| \| y \|} \tag{2.18}$$

Remark 2.9 1. By the Cauchy-Schwartz-Bunyakowski inequality, the right-hand side of Eq. 2.18 is always less than or equal to 1, and so the angle θ is well defined, $0 \leqslant \theta \leqslant \pi$, for every x and y different from 0.

2. If $X = R^3$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, then

$$\cos \theta = \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{(x_1^2 + x_2^2 + x_3^2)^{1/2} (y_1^2 + y_2^2 + y_3^2)^{1/2}}$$

This is a well-known relation in three-dimensional Euclidean space.

3. If $x \perp y$, then $\cos \theta = 0$, i.e., $\theta = \pi/2$. In view of this, orthogonal vectors are also called perpendicular vectors.

A well-known result of plane geometry is that the sum of the squares of the base and the perpendicular in a right-angled triangle is equal to the square of the hypotenuse, This is known as the Pythagorean theorem. Its infinite-dimensional analogue is as follows.

Theorem 2.8 Let X be an inne-product space and $x, y \in X$. Then for $x \perp y$, we have $\| x + y \|^2 = \| x \|^2 + \| y \|^2$.

Proof $\| x + y \|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle$ (by Definition 2.1 and Remark 2.1). Since $x \perp y$, $\langle x, y \rangle = 0$ and $\langle y, x \rangle = 0$ (by Definition 2.3 and Remark 2.8). Hence, $\| x + y \|^2 = \| x \|^2 + \| y \|^2$.

We recall (for details, see Appendix C) that an algebraic projection on a vector space X is a linear operator P on X into itself such that $P^2 = P$. The geometrical significance of this concept is as follows.

1. An algebraic projection determines a pair of subspaces M and N such that $X = M \oplus N$, where $M = \{P(x) \in X / x \in X\}$ and $N = \{x \in X / P(x) = 0\}$ are the range and null spaces of P.

2. A pair of subspaces M and N such that $X = M \oplus N$ determines an algebraic projection P whose range and null space are M and N.

Thus, the study of algebraic projection is equivalent to the study of pairs of subspaces which are disjoint and span X.

A *topological projection* on a Banach space is a continuous algebraic projection on a Banach space. Analogues of properties (1) and (2) stated above are true for topological projection. For details see Simmons[150], pp 236-238.] On an arbitrary Banach space, there may exist an algebraic projection. However, the existence of a topological projection is not guaranteed. The topological projection on a Hilbert space, called the *orthogonal projection* or *perpendicular projection* or projection, has some interesting properties. The existence of a topological projection on a Hilbert space is ensured by the projection theorem proved in the next subsection.

2.2.2 Projection Theorem

Theorem 2.9 (Projection theorem) If M is a closed subspace of a Hilbert space X, then

$$X = M \oplus M^{\perp}. \tag{2.19}$$

Remark 2.10 1. Theorem 2.9 implies that a Hilbert space is always rich in projections. In fact, for every closed subspace M of a Hilbert space X, there exists a topological projection on X whose range is M and whose null space is $M^{\perp}$.

2. Equation 2.19 means that every $z \in X$ is expressible uniquely in the form $z = x + y$ where $x \in M$ and $y \in M^{\perp}$. Since $M \cap M^{\perp} = \{0\}$, by Th. 2.7(2) in order to prove Th. 2.9, it is sufficient to show that $X = M + M^{\perp}$.

Equation 2.19 is called the *orthogonal decomposition* of Hilbert space X.

3. (i) Let $X = R^2$. Then Fig. 2.1 provides the geometric meaning of the orthogonal decomposition of R^2.

$$H = R^2,\ x \in R^2,\ x = y + z,\ y \in M,\ z \in M^{\perp}$$

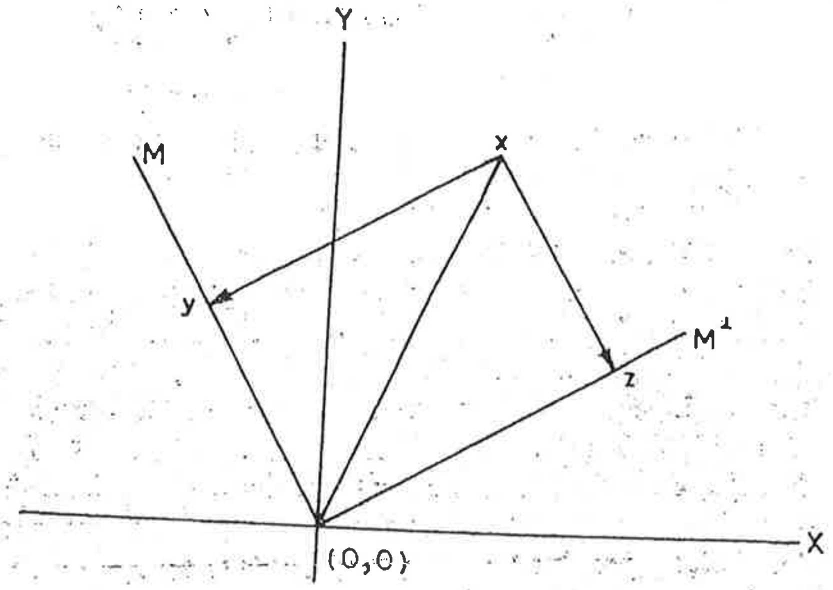

Fig. 2.1 Geometrical meaning of the orthogonal decomposition in R^2

(ii) Theorem 2.9 is not valid for inner-product spaces. (see Ex. 2.12(b)). We require the following results in the proof of Theorem 2.9.

Lemma 2.1 Let M be a closed convex subset of a Hilbert space X and $\rho = \inf_{x \in M} \| x \|$. Then there exists $x \in M$ such that $\| x \| = \rho$.

Lemma 2.2 Let M be a closed subspace of a Hilbert space X, let $x \notin M$ and let the distance between x and M be ρ, i.e., $\rho = \inf_{u \in M} \| x - u \|$.
Then there exists a unique vector $w \in M$ such that $\| x - w \| = \rho$.

Lemma 2.3 If M is a proper closed subspace of a Hilbert space X, there exists a nonzero vector u in X such that $u \perp M$.

Lemma 2.4 If M and N are closed subspaces of a Hilbert space X such that $M \perp N$, the subspace

$$M + N = \{x + y \in X / x \in M \text{ and } y \in N\} \text{ is also closed.}$$

Remark 2.11 Lemma 2.1 is true in the following general form. Let M be a closed convex subset of a Hilbert space X and for $x \in X$, let $\rho = \inf_{u \in M} \| x - u \|$. Then there exists a unique element $w \in M$ such that $\rho = \| x - w \|$. w is called the *projection* of x on M and we write $Px = w$.

Proof of Lemma 2.1 Since M is a convex subset of X, it is nonempty and $\alpha x + (1 - \alpha) y \in M$, for $\alpha = 1/2$ and every x, $y \in M$. By the definition of ρ, there exists a sequence of vectors $\{x_n\}$ in M such that $\| x_n \| \to \rho$. For $x = x_n$ and $y = y_m$, we have,

$$\frac{x_n + x_m}{2} \in M \quad \text{and} \quad \left\| \frac{x_n + x_m}{2} \right\| \geqslant \rho,$$

$$\therefore \quad \| x_n + x_m \| \geqslant 2\rho \tag{2.20}$$

By the parallelogram law for elements x_n and x_m,

$$\| x_n + x_m \|^2 + \| x_n - x_m \|^2 = 2 \| x_n \|^2 + 2 \| x_m \|^2$$

or
$$\| x_n - x_m \|^2 = 2 \| x_n \|^2 + 2 \| x_m \|^2 - \| x_n + x_m \|^2$$
$$\leqslant 2 \| x_n \|^2 + 2 \| x_m \|^2 - 4\rho^2 \quad \text{(By Eq. 2.20)}$$

Since $\| x_n \|^2 \to 2\rho^2$ and $\| x_m \|^2 \to 2\rho^2$, we have $\| x_n - x_m \|^2 \to 2\rho^2 + 2\rho^2 - 4\rho^2$
$= 0$ as $n, m \to \infty$.

Hence, $\{x_n\}$ is a Cauchy sequence in M. Since M is a closed subspace of a Banach space X, it is complete. (See Th. 1.3.) Hence, $\{x_n\}$ is convergent in M, i.e. there exists a vector x in M such that $\lim_{n \to \infty} x_n = x$. Since the norm is a continuous function, we have $\rho = \lim_{n \to \infty} \| x_n \| = \| \lim_{n \to \infty} x_n \| = \| x \|$.

Thus, x is an element of M with the desired property. Now, we show that x is unique. Let x' be another element of M such that $\rho = \| x' \|$. Since x,

$x' \in M$ and M is convex, $\frac{x+x'}{2} \in M$. By the parallelogram law for the elements $\frac{x}{2}$ and $\frac{x'}{2}$, we have

$$\left\| \frac{x}{2} + \frac{x'}{2} \right\|^2 + \left\| \frac{x}{2} - \frac{x'}{2} \right\|^2 = \frac{2}{4} \| x \|^2 + \frac{2}{4} \| x' \|^2$$

or
$$\left\| \frac{x+x'}{2} \right\|^2 = \frac{\| x \|^2}{2} + \frac{\| x' \|^2}{2} - \frac{\| x-x' \|^2}{2}$$
$$< \frac{\| x \|^2}{2} + \frac{\| x' \|^2}{2} = \rho^2$$

or
$$\left\| \frac{x+x'}{2} \right\| < \rho$$

which contradicts the definition of ρ. Hence, x is unique.

Proof of Lemma 2.2 The set $N = x + M = \{x + v / v \in M\}$ is a closed convex subset of X, and $\rho = \inf_{x+v \in N} \| 0 - (x+v) \|$ is the distance of $x+v$ from the origin 'o'. Since $-v \in M$ for all $v \in M$, $\rho = \inf_{v \in N} \| x - v \|$. By Lemma 2.1, there exists a unique vector $u \in N$ such that $\rho = \| u \|$. The vector $w = x - u = -(u - x)$ belongs to M. [As we have $M = N - x = \{z - x / z \in N; x \notin M\}$ and M is a subspace, therefore, $-(z-x) \in M$]. Thus, $\| x - w \| = \| u \| = \rho$. w is unique. For, if w is not unique, $w_1 \neq w$ is a vector in M such that $\| x - w_1 \| = \rho$. This implies that $u_1 = (x - w_1)$ is a vector in N such that $\| u_1 \| = \| x - w_1 \| = \rho$. This contradicts that u is unique. Hence, w is unique.

Proof of Lemma 2.3 Let $x \notin M$ and $\rho = \inf_{v \in M} \| x - v \|$, the distance from x to M. By Lemma 2.2, there exists a unique element $w \in M$ such that $\| x - w \| = \rho$.

Let $u = x - w$. $u \neq 0$ as $\rho > 0$. (If $u = 0$, then $x - w = 0$ and $\| x - w \| = 0$ implies that $\rho = 0$.)

Now, we show that $u \perp M$. For this, we show that for arbitrary $y \in M$, $\langle u, y \rangle = 0$.

For any scalar α, we have $\| u - \alpha y \| = \| x - w - \alpha y \| = \| x - (w + \alpha y) \|$. Since M is a subspace, $w + \alpha y \in M$ whenever $w, y \in M$. Thus, $w + \alpha y \in M$ implies that $\| u - \alpha y \| \geq \rho = \| u \|$ or $\| u - \alpha y \|^2 - \| u \|^2 \geq 0$ or $\langle u - \alpha y, u - \alpha y \rangle - \| u \|^2 \geq 0$.

$$\because \quad \langle u - \alpha y, u - \alpha y \rangle = \langle u, u \rangle - \alpha \langle y, u \rangle - \bar{\alpha} \langle u, y \rangle + \alpha\bar{\alpha} \langle y, y \rangle$$
$$= \| u \| - \bar{\alpha} \langle u, y \rangle - \alpha \langle y, u \rangle + | \alpha |^2 \langle y, y \rangle,$$

we have,

$$-\bar{\alpha} \langle u, y \rangle - \alpha \overline{\langle u, y \rangle} + | \alpha |^2 \| y \|^2 \geq 0 \tag{2.21}$$

By putting $\alpha = \beta \langle u, y \rangle$ in Eq. 2.21, β being an arbitrary real number, we get

$$-2\rho \, | \langle u, y \rangle |^2 + \beta^2 | \langle u, y \rangle |^2 \| y \|^2 \geq 0 \tag{2.22}$$

If we put $a = |\langle u, y\rangle|^2$ and $b = \|y\|^2$ in Eq. 2.22, we obtain

$$-2\beta a + \beta^2 ab \geqslant 0$$

or

$$\beta a(\beta b - 2) \geqslant 0 \;\forall \text{ real } \beta \tag{2.23}$$

If $a > 0$, Eq. 2.23 is false for all sufficiently small positive β. Hence, a must be zero, i.e., $a = |\langle u, y\rangle|^2 = 0$ or $\langle u, y\rangle = 0$ y $\forall$ $y \in M$.

This gives the desired result.

Proof of Lemma 2.4 It is a well-known result of vector spaces (see Appendix E) that $M+N$ is a subspace of X. We show that it is closed, i.e., every limit point of $M+N$ belongs to it. Let z be an arbitrary limit point of $M+N$. Then there exists a sequence $\{z_n\}$ of points of $M+N$ such that $z_n \to z$. (See Theorem C.2 of Appendix C.) $M \perp N$ implies that $M \cap N = \{0\}$. So, every $z_n \in M+N$ can be written uniquely in the form $z_n = x_n + y_n$, where $x_n \in M$ and $y_n \in N$.

By the Pythagorean theorem for elements $(x_m - x_n)$ and $(y_m - y_n\}$, we have

$$\begin{aligned}\|z_m - z_n\|^2 &= \|(x_m - x_n) + (y_m - y_n)\|^2 \\ &= \|x_m - x_n\|^2 + \|y_m - y_n\|^2\end{aligned} \tag{2.24}$$

(It is clear that $(x_m - x_n) \perp (y_m - y_n) \forall m, n$.) Since $\{z_n\}$ is convergent, it is a Cauchy sequence and so $\|z_m - z_n\|^2 \to 0$. In view of it, from Eq. 2.24, we see that $\|x_m - x_n\| \to 0$ and $\|y_m - y_n\| \to 0$ as $m, n \to \infty$. Hence, $\{x_m\}$ and $\{y_n\}$ are Cauchy sequences in M and N respectively. Being closed subspaces of a complete space, M and N are also complete. Hence, $\{x_m\}$ and $\{y_n\}$ are convergent in M and N respectively, say $x_m \to x \in M$ and $y_n \to y \in N$. $x+y \in M+N$ as $x \in M$ and $y \in N$. Then,

$$\begin{aligned}z = \lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + y_n) &= \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n \\ &= x + y \in M+N\end{aligned}$$

This proves that an arbitrary limit point of $M+N$ belongs to it and so it is closed.

Proof of Theorem 2.9 Since M is a closed subspace of X, by Th. 2.7 (1), $M^\perp$ is also a closed subspace of X. By choosing $N = M^\perp$ in Lemma 2.4, we find that $M + M^\perp$ is a closed subspace of X. First, we want to show that $X = M + M^\perp$. Let $X \neq M + M^\perp$, i.e., $M + M^\perp$ is a proper closed subspace of X. Then by Lemma 2.3, there exists a nonzero vector u such that $u \perp M + M^\perp$. This implies that $\langle u, x+y\rangle = 0$, $\forall$ $x \in M$ and $y \in M^\perp$. If we choose $y = 0$, then $\langle u, x\rangle = 0$ $\forall$ $x \in M$, i.e., $u \in M^\perp$. On the other hand, if we choose $x = 0$ then $\langle u, y\rangle = 0$ $\forall$ $y \in M^\perp$ i.e., $u \in M^{\perp\perp}$. (Since M and $M^\perp$ are subspaces this choice is possible.) Thus, $u \in M^\perp \cap M^{\perp\perp}$. By Th. 2.7(2) for $A = M^\perp$, we obtain $u = 0$. This is a contradiction as $u \neq 0$. Hence, our assumption is false and $X = M + M^\perp$. In view of Remark 2.10(2); the theorem is proved.

Remark 2.12 Sometimes, the following statements is also added in the statement of the projection theorem whose proof we shall give in Example 2.29. "Let M be a closed subspace of a Hilbert space X, then $M = M^{\perp\perp}$".

Remark 2.13 1. Let $X = L_2(-1, 1)$. Then $X = M \oplus M^{\perp}$,

where $M = \{f \in L_2(-1, 1) / f(-t) = f(t) \forall t \in (-1, 1)$, i.e., the space of even functions$\}$

$M^{\perp} = \{f \in L_2(-1, 1) / f(-t) = -f(t) \forall t \in (-1, 1)$, i.e., the space of odd functions$\}$.

2. Let $X = L_2[a, b]$; for $c \in [a, b]$, let

$$M = \{f \in L_2[a, b] / f(t) = 0 \text{ almost everywhere in } (a. c)\}$$

and $$M^{\perp} = \{f \in L_2[a, b] / f(t) = 0 \text{ almost everywhere in } (c, b)\}.$$

Then, $X = M \oplus M^{\perp}$.

Remark 2.14 The orthogonal decomposition of a Hilbert space, i.e., Th. 2.9, has been proved quite useful in potential theory. [For details, see Taylor[160] and Weyl[175].] The applications of the results concerning orthogonal decomposition of Hilbert spaces can be found in spectral decomposition theorems which deal with the representation of operators on Hilbert spaces. For example, for a bounded self-adjoint operators, T, $\langle Tx, y \rangle$ is represented by an ordinary Riemann-Stieltjes integral. For details, see Kreyszig[96], Naylor and Sell[121], Taylor[160], and solutions of Ex. 8.1 and 8.2.

2.3 ORTHONORMAL SYSTEMS AND FOURIER EXPANSION

2.3.1 Definitions, Examples and Gram-Schmidt Orthogonalization Process

Definition 2.5 Let X be a Hilbert space and $\{\phi_n\}$ be a sequence of elements of X. Then:

1. $\{\phi_n\}$ is called an *orthonormal system* in the Hilbert space X if

$$\langle \phi_n, \phi_m \rangle = 0 \quad \text{if} \quad m \neq n$$
$$= 1 \quad \text{if} \quad m = n.$$

2. For any $f \in X$, $\alpha_p = \langle f, \phi_p \rangle$, $p = 0 \pm 1, \pm 2, \ldots$ is called the p^{th} *Fourier coefficient* with respect to orthonormal system $\{\phi_n\}$.

$\sum_{-\infty}^{\infty} \langle f, \phi_p \rangle \phi_p$ is called the *Fourier series or Fourier expansion* of f with respect to the orthonormal system $\{\phi_n\}$.

3. The orthonormal family $\{\phi_n\}$ is said to be *complete* if there exists no other orthonormal family containing it.

4. The orthonormal family $\{\phi_n\}$ is said to be an *orthonormal basis* or *closed orthonormal system* if the sum of the Fourier series of f with respect to $\{\phi_n\}$ is equal to $f \forall f \in X$, i.e., $f = \sum_{p=1}^{\infty} \langle f, \phi_p \rangle \forall f \in X$.

Remark 2.15 1. If $X = \{0\}$, then X contains no orthonormal system.

2. An orthonormal system $\{\phi_n\}$ is complete iff and for any x such that $x \perp \{\phi_n\}$, x must be zero.

3. Every nonzero Hilbert space contains a closed orthonormal set.

4. Every orthonormal system is closed iff it is complete.

EXAMPLE 2.9 Consider the Hilbert space $L_2(0, 2\pi)$ of complex-valued functions. (See Example 2.5) $\frac{e^{inx}}{\sqrt{2\pi}}$, $n = 0, \pm 1, \pm 2 \ldots$ is an orthonormal system as

$$\langle e^{imx}, e^{inx} \rangle = \int_0^{2\pi} \frac{e^{imx}}{\sqrt{2\pi}} \frac{\overline{e^{inx}}}{\sqrt{2\pi}} dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{imx} e^{-inx} dx = 0 \qquad \text{if } m \neq n$$

$$= 1 \qquad \text{if } m = n$$

$$\alpha_n = c_n = \langle f, \frac{e^{inx}}{\sqrt{2\pi}} \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx, \; n = 0, \; n = 0, \pm 1, \pm 2 \ldots$$

are the Fourier coefficients of $f \in L_2(0, 2\pi)$ with respect to the orthonormal system $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}$. $\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}$ is the Fourier series of f.

This orthonormal system is known as *complex trigonometric system*. This system has been studied in great detail. (See, for example, Zygmund[181].)

EXAMPLE 2.10 If we consider $L_2(0, 2\pi)$, $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt\right\}$ $n = 1, 2, 3 \ldots$, is an orthonormal system in $L_2(0, 2\pi)$, i.e.,

$$\phi_n = \frac{1}{\sqrt{\pi}} \cos\left(\frac{n}{2} t\right) \qquad \text{when } n \text{ is an even positive integer}$$

$$= \frac{1}{\sqrt{\pi}} \sin\left(\frac{n+1}{2} t\right) \qquad \text{when } n \text{ is an odd positive integer}$$

We now define, for $f \in L_2(0, 2\pi)$, $\alpha_n = \langle f, \phi_n \rangle$, i.e.

$$\alpha_0 = \langle f, \phi_0 \rangle = \int_0^{2\pi} f(t) \frac{1}{\sqrt{2\pi}} dt = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t)\, dt$$

$$\alpha_1 = \langle f, \phi_1 \rangle = \int_0^{2\pi} f(t) \phi_1(t)\, dt$$

$$= \int_0^{2\pi} f(t) \frac{1}{\sqrt{\pi}} \sin t \, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t) \sin t \, dt$$

$$\alpha_2 = \langle f, \phi_2 \rangle = \int_0^{2\pi} f(t)\phi_2(t) \, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t) \cos t \, dt$$

$$\alpha_3 = \langle f, \phi_3 \rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t) \sin 2t \, dt$$

$$\alpha_4 = \langle f, \phi_4 \rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t) \cos 2t \, dt$$

Consider now the ordinary Fourier coefficients

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t) \, dt$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos \left(\frac{n}{2} t\right) dt \quad \text{where } n \text{ is an even positive integer}$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin \left(\frac{n+1}{2} t\right) dt \quad \text{where } n \text{ is an odd positive integer}$$

Then we have following relationships.

$$\alpha_0 = a_0 \sqrt{\frac{\pi}{2}}$$

$$\alpha_1 = b_1 \sqrt{\pi}$$

$$\alpha_2 = a_2 \sqrt{\pi}$$

$$\alpha_3 = b_3 \sqrt{\pi}$$

$$\alpha_4 = a_4 \sqrt{\pi}$$

$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is known as the *trigonometric Fourier series* of $f \in L_2(0, 2\pi)$.

This has been extensively studied. [See Bari[15] and Zygmund[181].]

EXAMPLE 2.11 Let us define a sequence of functions $\phi_0(t), \phi_1(t), \ldots \phi_n(t), \ldots$ which satisfy the following conditions.

$$\varphi_0(t) = 1 \qquad \text{if } 0 \leqslant t < \frac{1}{2}$$

$$\varphi_0(t) = -1 \qquad \text{if } \frac{1}{2} \leqslant t < 1$$

$$\phi_0(t+1) = \phi_0(t)$$
$$\phi_n(t) = \phi_0(2^n t)$$

where $n = 1, 2, 3, \ldots$

The functions $\phi_n(t)$ are called *Rademacher's functions.* This system of functions is orthogonal but not complete.

Now, we define a system of functions in terms of Rademacher systems as follows. $\psi_0(t) = 1$, $\psi_n(t) = \phi_{n_1}(t)\phi_{n_2}(t) \ldots \phi_{n_v}(t)$, $0 \leqslant t \leqslant 1$, for $n = 2^{n_1} + 2^{n_2} + \ldots 2_{n_v}$, where the nonnegative integers n_i are uniquely determined by the inequalities $n_{i+1} < n_i$.

It is clear from the definition that $\psi_2^m(t) = \phi_m(t)$.

The system of functions $\{\psi_n(t)\}$, defined above, is called the *system of Walsh functions*. This is a complete orthonormal system in the Hilbert space of real-valued functions $L_2(0, 1)$. $a_n = \langle f, \psi_n \rangle = \int_0^1 f(t)\psi_n(t)\, dt$, where $n = 0, 1, 2, 3, \ldots$, are Walsh Fourier coefficients of $f \in L_2(0, 1)$ and $\sum_0^\infty a_n\psi_n(t)$ is the Walsh Fourier series of f.

The concept of Walsh functions and its Fourier expansion was given by JL Walsh in 1923. A major breakthrough came in the researches of NJ Fine in 1946. He introduced the concept of the dyadic group G which played a significant role in the development of this theory. The concept of the dyadic derivative, investigated in early seventies by Gibbs, Millard, Butzer and Wagner, was of vital importance for many applications. In the last decade, this has been a favourite topic of research for electrical and electronic engineers; especially in the USA, UK, FRG and India. For details we refer to Beauchamp[16], Harmuth[72], Siddiqi A H[144] Siddiqi M U[146]. This is one of the most interesting topics for theoretical and applied researches of our time.

Theorem 2.10 (Gram-Schmidt orthogonalization process) Let X be an inner-product space and $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of elements of X. Then there exists an orthonormal system $\{\phi_i\}$ $i = 1, 2, \ldots, n$ such that for each n the subspace of X spanned by $\{x_1, x_2, \ldots, x_n, \ldots\}$ is the same as the space of X spanned by $\{\phi_i\}$.

Proof We prove it by induction. In other words, we show that the statement is true for $n = 1$ and true for n whenever true for $n - 1$. This will imply that it is true for all natural numbers n.

Let $n = 1$. Then $x_1 \neq 0$. Now, $\phi_1 = \frac{x_1}{\| x_1 \|}$ and $\| \phi_1 \| = \langle \frac{x_1}{\| x_1 \|}, \frac{x_1}{\| x_1 \|} \rangle = \frac{1}{\| x_1 \|} \langle x_1, x_1 \rangle = 1$ (using results of Section 2.1).

Since $\| x_1 \| \phi_1 = x_1$, the set of linear combinations of x_1 is the same as the set of linear combinations of $\| x_1 \| \phi_1$. Hence, the subspace spanned by x_1 is the same as the subspace spanned by $\| x_1 \| \varphi_1$. Thus, the theorem is true for $n = 1$.

Suppose that theorem is true for $n-1$. We now construct an nth vector ϕ_n with the same property. For this, we consider the vector $u = x_n - \sum_{i=1}^{n} \langle x_n, \phi_i \rangle \phi_i$ and $1 \leq j \leq n-1$. Then, $\langle u, \phi_j \rangle = \langle x_n - \sum_{i=1}^{n} \langle x_n, \phi_i \rangle \phi_i, \phi_j \rangle$.

By the application of the results in Sec. 2.1, we obtain

$$\langle u, \phi_j \rangle = \langle x_n, \phi_j \rangle - \sum_{i=1}^{n} \langle x_n, \phi_i \rangle \langle \phi_i, \phi_j \rangle$$

$$= \langle x_n, \phi_j \rangle - \sum_{i=1}^{n} \langle x_n, \phi_i \rangle \langle \phi_i, \phi_j \rangle$$

$\because$
$$\langle \phi_i, \phi_j \rangle = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j,$$

the above relation gives us

$$\langle u, \phi_j \rangle = \langle x_n, \phi_j \rangle - \langle x_n, \phi_i \rangle = 0$$

Hence, $u \perp \phi_j$, $j = 1, 2, \ldots, n-1$.

$u \neq 0$, otherwise $x_n = \sum_{i=1}^{n} \langle x_n, \phi_i \rangle \phi_i$, i.e. x_n is a linear combination of $\varphi_1, \phi_2, \ldots, \phi_{n-1}$ and since the theorem is true for $n-1$; x_n is also linear combination of $x_1, x_2, \ldots, x_{n-1}$, which is a contradiction to the fact that $\{x_1, x_2, \ldots, x_{n-1}, x_n, \ldots\}$ is a linearly independent set.

Now, choose $\phi_n = \dfrac{u}{\| u \|}$. Thus $\{\phi_i\}$, $i = 1, 2, \ldots n$ is an orthonormal system.

Since
$$\phi_p = \frac{x_p - \sum_{i=1}^{p-1} \langle x_p, \phi_i \rangle \phi_i}{\| x_p - \sum_{i=1}^{p-1} \langle x_p, \phi_i \rangle \phi_i \|} \quad \text{for } p = 1, 2, 3, \ldots, n;$$

every x_p can be expressed as the linear combination of φ_p. Hence, the subspace spanned by $\{x_1, x_2, \ldots, x_n\}$ is the same as the subspace spanned by $\{\phi_i\}$ $i = 1, 2, \ldots, n$.

This proves the theorem.

2.3.2 Bessel's Inequality

Theorem 2.11 (Bessel's inequality). Let $\{\phi_i\}$ $i = 1, 2, \ldots, n$ be an orthonormal system of vectors in a Hilbert space X.

Then, for any $f \in X$, $\sum_{i=1}^{n} | \langle f, \phi_i \rangle |^2 \leq \| f \|^2$.

Proof Let $g = f - \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i$. Then for any j, $1 \leq j \leq n$, we have

$$\langle g, \phi_j \rangle = \langle f - \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i, \phi_j \rangle$$

$$= \langle f, \phi_j \rangle - \sum_{i=1}^{n} \langle f, \phi_i \rangle \langle \phi_i, \phi_j \rangle$$

by using the properties of the inner product.

$$\langle \varphi_i, \varphi_j \rangle = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j$$

The above relation gives us $\langle g, \varphi_j \rangle = \langle f, \varphi_i \rangle - \langle f, \varphi_i \rangle = 0$.

Hence $g \perp \varphi_j$ for $1 \leqslant j \leqslant n$, and the vectors g, $\langle f, \varphi_1 \rangle \varphi_1$, $\langle f, \varphi_2 \rangle \varphi_1$, $\ldots$, $\langle f, \varphi_n \rangle \varphi_n$ form an orthonormal system.

By the Pythagorean theorem, we have $\| f \|^2 = \| g + \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i \|^2 = \| g \|^2 + \| \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i \|^2$.

Applying Pythagorean theorem for n elements, we have

$$\| \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i \|^2 = \sum_{i=1}^{n} \| \langle f, \varphi_i \rangle \varphi_i \|^2$$
$$= \sum_{i=1}^{n} | \langle f, \varphi_i \rangle |^2 \| \varphi_i \|^2$$
$$= \sum_{i=1}^{n} | \langle f, \varphi_i \rangle |^2$$

$$\because \quad \| \varphi_i \|^2 = \langle \varphi_i, \varphi_i \rangle = 1$$

(Let $x_1, x_2, \ldots, x_n$ be n orthogonal elements. Then $\| \sum_{i=1}^{n} x_i \|^2 = \sum_{i=1}^{n} \| x_i \|^2$ follows by induction from the Pythagorean theorem.)

Thus, $\| f \|^2 = \| g \|^2 + \sum_{i=1}^{n} | \langle f, \varphi_i \rangle |^2$.

Since $\| g \|^2 \geqslant 0$, $\sum_{i=1}^{n} | \langle f, \varphi_i \rangle |^2 \leqslant \| f \|^2$. This proves the desired result.

Corollary 2.1 1. For X and $\{\varphi_n\}$ as in Example 2.10 we have $\sum_{n=n}^{\infty} | \alpha_n |^2 < \infty$.

2. If a_n's and b_n's are ordinary trigonometric Fourier coefficients, then $\lim_{n \to \infty} a_n = 0$, and $\lim_{n \to \infty} b_n = 0$.

This result is known as the *Riemann-Lebesgue theorem*.

Proof 1. Since $f \in L_2(0, 2\pi)$ and $\langle f, \varphi_i \rangle = \alpha_i$, the Bessel's inequality gives us $\sum_{n=0}^{\infty} | \alpha_n | \leqslant \| f \|^2 < \infty$.

2. By the relationships between a_n's, b_n's and α_n's given in Example 2.10 we have

$$\sum_{n=0}^{\infty} | \alpha_n | = a_0^2 \frac{\pi}{2} + \pi\left(\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right)$$
$$\leqslant \| f \|^2 = \int_0^{2\pi} | f(t) |^2 \, dt < \infty$$

i.e., $\sum_{n=1}^{\infty}(a_n^2+b_n^2) \leqslant \infty$, which implies that

$$\lim_{n\to\infty}(a_n^2+b_n^2)=0$$

or

$$\lim_{n\to\infty} a_n^2=0$$

and

$$\lim_{n\to\infty} b_n^2=0$$

$\therefore$

$$\lim_{n\to\infty} a_n=0$$

and

$$\lim_{n\to\infty} b_n=0$$

Theorem 2.12 For an orthonormal system $\{\varphi_i\}$, $i=1, 2, \ldots, n$, in a Hilbert X, for each $f\in X$, the following relation holds.

$$\|f\|^2=\|\sum_{i=1}^{n} |\langle f, \varphi_i\rangle|^2 \tag{2.25}$$

iff

$$f=\sum_{i=1}^{n}\langle f, \varphi_i\rangle\varphi_i$$

Proof Suppose that $\{\varphi_i\}$ is a closed orthonormal system in X i.e.

$$f=\sum_{i=1}^{n}\langle f, \varphi_i\rangle\varphi_i.$$

Then

$$\|f\|^2=\|\sum_{i=1}^{n}\langle f, \varphi_i\rangle\varphi_i\|^2$$

$$=\sum_{i=1}^{n}|\langle f, \varphi_i\rangle| \, \|\varphi_i\|^2$$

We get the above relation by applying the Pythagorean theorem for n orthogonal elements, namely

$$\langle f, \varphi_i\rangle\varphi_1, \langle f, \varphi_2\rangle\varphi_2, \ldots, \langle f, \varphi_n\rangle\varphi_n.$$

As $\|\varphi_i\|=1$, we have $\|f\|^2=\sum_{i=1}^{n}|\langle f, \varphi_i\rangle|^2$.

Conversely, if $\|f\|^2=\sum_{i=1}^{n}|\langle f, \varphi_i\rangle|^2$, as shown in the proof of the Bessel's inequality, we have $\|f\|^2=\|g\|^2+\sum_{i=1}^{n}|\langle f, \varphi_i\rangle|^2$, where $g=f-\sum_{i=1}^{n}\langle f, \varphi_i\rangle\varphi_i$. This implies that $\|g\|^2=0$, i.e., $g=f-\sum_{i=1}^{n}\langle f, \varphi_i\rangle\varphi_i=0$ or $f=\sum_{i=1}^{n}\langle f, \varphi_i\rangle\varphi_i$.

Remark 2.16 The relation $\|f\|^2=\sum_{i=1}^{\infty}|\langle f, \varphi_i\rangle|^2$ is known as *Parseval's identity*.

2. Parseval's identity is valid iff the orthonormal system $\{\varphi_i\}$ is closed.

2.4 DUALITY AND REFLEXIVITY

2.4.1 Riesz Representation Theorem

In this section, we prove a theorem which gives the representation of a bounded linear functional defined on a Hilbert space and with the help of it the relation between a Hilbert space and its dual are studied.

Theorem 2.13 (Riesz representation theorem). If f is a bounded linear functional on a Hilbert space X, there exists a unique vector $y \in X$ such that $f(x) = \langle x, y \rangle \; \forall x \in X$, and $\| f \| = \| y \|$.

Proof 1. In the first place, we prove that there exists an element y such that $f(x) = \langle x, y \rangle \; \forall x \in X$.

(a) If $f = 0$, then $f(x) = 0 \; \forall x \in X$. Therefore, $y = 0$ is the vector for which $\langle x, y \rangle = 0 \; \forall x \in X$. (We have seen that $\langle x, 0 \rangle = 0$.) Thus, the existence of vector y is proved when $f = 0$.

(b) Let $f \neq 0$. By proposition 1.1(4), the null space N of f is a proper closed subspace of X and by Lemma 2.3, there exists a nonzero vector $u \in X$ such that $u \perp N$. We show that if α is a suitably chosen scalar, $y = \alpha u$ satisfies the condition of the theorem.

(c) If $x \in N \subset X$, then whatever be α, $f(x) = 0 = \bar{\alpha} \langle x, u \rangle$ as $u \perp N$. Thus $f(x) = \langle x, \alpha u \rangle$. Hence the existence of $y = \alpha u$ is proved for all $x \in N$.

(d) Since $u \perp N$, $u \notin N$; let $x = u \in X - N$. If $f(u) = \langle u, \alpha u \rangle = \bar{\alpha} \| u \|^2$, then

$$\bar{\alpha} = \frac{f(x)}{\| u \|^2}.$$

Therefore, for $\alpha = \dfrac{\overline{f(x)}}{\| u \|^2}$, the vector $y = \alpha u$ satisfies the condition of the theorem in this case, i.e. $f(u) = \langle u, \alpha u \rangle$.

(e) Since $u \notin N$, $f(u) \neq 0$ and so $\dfrac{f(x)}{f(u)}$ is defined for any $x \in X$. Consider $x - \beta u$, where $\beta = \dfrac{f(x)}{f(u)}$. Then $f(x - \beta u) = f(x) - \beta f(u) = 0$. This implies that $x - \beta u \in N$.

Every $x \in X$ can be written as $x = x - \beta u + \beta u$. Therefore, for each $x \in X$,

$$\begin{aligned} f(x) &= f(x - \beta u + \beta u) = f(x - \beta u) + f(\beta u) \\ &= f(x - \beta u) + \beta f(u) \qquad (f \text{ is linear}) \\ &= \langle x - \beta u, \alpha u \rangle + \beta \langle u, \alpha u \rangle \end{aligned}$$

by (c) and (d), where $\alpha = \dfrac{\overline{f(x)}}{\| u \|^2}$. $\left(\text{Since } x - \beta u \in N, \text{ by (c)}, f(x - \beta u) = \langle x - \beta u, \alpha u \rangle, \text{ for every } \alpha \text{ and so for } \alpha = \dfrac{\overline{f(x)}}{\| u \|^2}. \right)$

This gives us $f(x) = \langle x - \beta u, \alpha u \rangle + \langle \beta u, \alpha u \rangle = \langle x, \alpha u \rangle$. Thus, for an arbitrary $x \in X$, there exists a vector $y = \alpha u$ such that $f(x) = \langle x, y \rangle$.