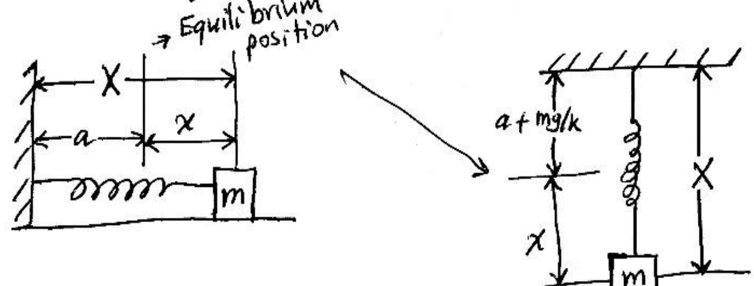
Chapter Three
(Oscillations)

Introduction:

Everywhere around us we see systems engaged in a periodic dance: the swaying تأرجح of a tree in the wind, the small oscillations of a ~~tree~~ pendulum clock, a child playing on a swing, the rise and fall of the tides المد والجزر. The essential أساسي feature that all these phenomena have in common is periodicity دورية, a pattern of movement or displacement that repeats itself over and over again.

3-I: Linear Restoring Force. Harmonic Motion



* a particle of mass (m) attached to a spring (a) horizontally (b) vertically.

واحدة من الحالات الأكثر أهمية في الحركة الخطية مناحيه نظر عملية علاوة على وجيه
النظام النظري هي تلك الناتجة بواسطة القوة المعيدة الخطية linear restoring force.

This is a force whose magnitude is proportional to the displacement of a particle from some equilibrium position and whose direction is always opposite to that of the displacement

$$F = -k(X-a) = -kx \qquad \text{Hook's Law}$$

where:

X - total length

a - unstretched (zero load) length of the spring.

x - is the displacement of the spring from its equilibrium length

k - the proportionality ~~of~~ constant called stiffness

The total force acting on the particle is:

the posative direction is (downward motion)

$$F = -k(X-a)+mg$$

The diff. equation of motion for the horizontal case is:

$$F = -K X \quad \& \quad F = m\ddot{X}$$

$$-kX = m\ddot{X}$$ ---- ① the diff. eq. of the harmonic oscillator of linear oscillator

$$\Rightarrow \boxed{m\ddot{X} + kX = 0}$$ linear diff. eq. with const. coef.

where; m & k are two const. mass & stifness const.

To solve the above equation we shall employ the trial method in which the function (Ae^{qt}) is the trial solution, where (q) is a constant to be determined.

So;

$$x = Ae^{qt} \longrightarrow \dot{x} = Aqe^{qt} \longrightarrow \ddot{X} = Aq^2e^{qt} \quad ---②$$

sub. eqs. ② in eq ① we get:

$$m\frac{d^2}{dt^2}(q^2e^{qt}) + KAe^{qt} = 0$$

$$m\frac{d^2}{dt^2}(Ae^{qt}) + KAe^{qt} = 0$$

$$[mAq^2e^{qt} + KAe^{qt} = 0] \quad \div Ae^{qt}$$

$$mq^2 + k = 0 \longrightarrow q^2 = \frac{-k}{m} \longrightarrow q = \mp\sqrt{\frac{-k}{m}}$$

$$= \mp i\sqrt{\frac{k}{m}} = \mp iW_0$$

where $i = \sqrt{-1}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

in $x = Ae^{\mp i\omega_0}$

for a linear diff. eqs., solutions are additive. so the general solution of eq① is:

$$x = A_+ e^{+i\omega_0 t} + A_- e^{-i\omega_0 t} \quad \text{---- I}$$

where: $e^{+i\omega_0 t} \rightarrow$ is an outgoing wave.

$e^{-i\omega_0 t} \longrightarrow$ is an Incoming wave.

using Uler formula:

$$e^{ix} = \cos x + i \sin x \ , \quad e^{-ix} = \cos x - i \sin x$$

$$x = A^+(\cos\omega_0 t + i \sin\omega_0 t) + A^-(\cos\omega_0 t - i \sin\omega_0 t)$$

$$= A^+\cos\omega_0 t + iA^+\sin\omega_0 t + A^-\cos\omega_0 t - iA^-\sin\omega_0 t$$

$$= \underbrace{2A^+}_{a}\cos\omega_0 t + \underbrace{2A^-}_{b}\sin\omega_0 t$$

$$x = a\cos\omega_0 t + b\sin\omega_0 t \quad \text{---- II} \quad \left[\times \frac{\cos\omega_0 t}{\cos\omega_0 t}\right] \text{ or } \left[\times \frac{\sin\omega_0 t}{\sin\omega_0 t}\right]$$

$$= a\cos\omega_0 t + b$$

$$\boxed{x = A\sin(\omega_0 t + \theta_0)}$$

$$\boxed{x = \underline{A}\cos(\omega_0 t + \theta_0)} \quad \text{---- III}$$

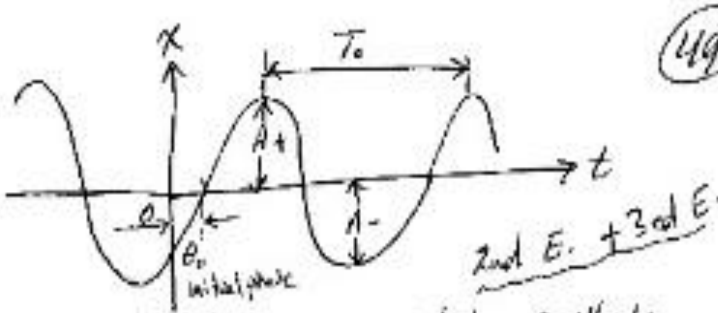
where; $A = (a^2 + b^2)^{1/2}$, $\boxed{\theta_0 = -\tan^{-1}\left(\frac{b}{a}\right)}$ (Initial phase)

A - is amplitude

ω_0 - angular freq.

where all three above equations are solutions of eq①

$$T_o = \frac{2\pi}{\omega_o} = 2\pi\sqrt{\frac{m}{k}}$$



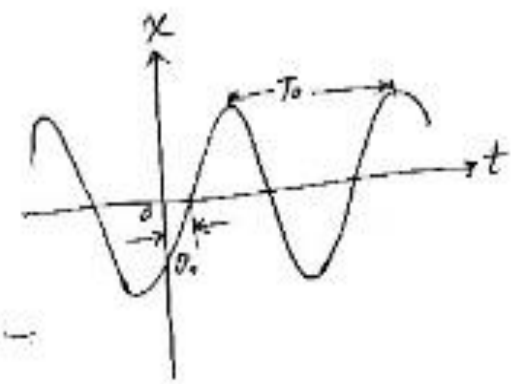
① period: the time for which the product (ωt) increases by just (2π)

The motion is a sinusoidal oscillation of the displacement x

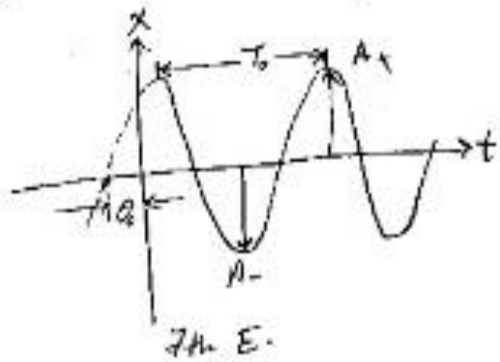
② the linear frequency of oscillation (f_o) is defined as the number of cycles in unit time, therefore:

(radians per second) $\rightarrow \omega_o = 2\pi f_o$

(circles per second) $f_o = \frac{1}{T_o} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$



$x = A \sin(\omega_o t + \theta_o)$
equation of simple harmonic oscillator



7th E.

$x = A \cos(\omega_o t + \theta_o)$
simple harmonic oscillator

3-2 Energy considerations in Harmonic Motion:

اعتبارات الطاقة في الحركة التوافقية

consider a particle moving under a restoring force $F=-kx$. Let us calculate the worke (W) done by an external force (F_a) in moving the particle from the equiliberium position ($x=0$) to some position x.

$$F=-kx \quad \text{---} (1)$$

$$F_a=-F$$

$$=-(-kx)=kx$$

$$W=\int F_a(x)\,dx = \int_0^x (kx)\,dx = \frac{1}{2}kx^2 \quad \text{---} (2)$$

تمثل معادلة (2) الشغل المبذول من قبل (F_a) على النابض لتحريكه خارج إزاحة الاتزان. فإذا ما أُزيلت هذه القوة الشغل المبذول ومثبتة النابض عند موضع معين (الرجوع إلى موضع الاتزان (a)). فان هذا الشغل سيبقى مخزونا في النابض على شكل طاقة كامنة تحسب كالاتي:

$$V(x)=W=\frac{1}{2}kx^2=E_p \quad \text{---} (3)$$

Total Spring energy:

$$\boxed{E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2} \quad \text{---} (4)$$

We can now solve for the velocity (as a function of displacement):

$$\dot{x}^2=\frac{2}{m}\left[E-\frac{1}{2}kx^2\right]$$

$$\therefore \boxed{\dot{x}=\left[\frac{2E}{m}-\frac{kx^2}{m}\right]^{1/2}} \quad \text{---} (5)$$

This can be integrated to give (t) as a function of x:

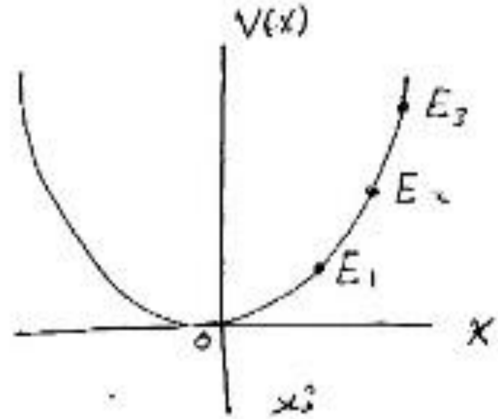
$$\dot{x}=\frac{dx}{dt}=\sqrt{\frac{2E}{m}-\frac{kx^2}{m}} \longrightarrow \boxed{t=\int_0^x \frac{dx}{\text{[illegible]}}}$$

(51) لغرض حساب التكامل اعلاه يجب توفير مشتقة المعادلة والذي هي ؛

$\frac{K}{m}\, 2X$

$$\boxed{t = \sqrt{\frac{m}{k}}\, \cos^{-1}\left(\frac{x}{A}\right) + C}$$

where: $A = \sqrt{\frac{2E}{k}}$



للمتذبذب البسيط يمتلك قيم طاقة متصلة لكل الازاحة القراءة (x) عن موضع الاتزان (0) يميناً او يساراً والتي هي عبارة عن مجموع طاقة حركية ما لم يصل الى المعظم ازاحة (وهي السعة) وهناك ستكون كل طاقته كامنة فقط . في حين في نقطة الاتزان تكون الطاقة الكلية صفر . وهذا هو التفسير الكلاسيكي لتقاسم الطاقة في المتذبذب التوافقي البسيط

(1) $$E = \underbrace{\frac{1}{2} m v^2_{max.}}_{\text{عند نقطة الاتزان}} = \underbrace{\frac{1}{2} k A^2}_{\text{في القمة}}$$

$$\therefore v^2_{max.} = \frac{2E}{m}$$

We have: $A^2 = \frac{2E}{k} \rightarrow E = \frac{kA^2}{2}$

$$\therefore v^2_{max} = \frac{\cancel{2}\,\frac{kA^2}{\cancel{2}}}{m} = \frac{kA^2}{m}$$

$$\Rightarrow \boxed{V_{max.} = \sqrt{\frac{k}{m}}\, A}$$ Maximum velocity for the harmonic oscillator.

(2) Let: $\underset{\text{lower point}}{E} = \underset{\text{higher point}}{E} \longrightarrow \cancel{\frac{1}{2}} m v^2_{max.} = \cancel{\frac{1}{2}} kA^2 \Rightarrow A^2 = \frac{m}{k} v^2_{max}$

$$\therefore \boxed{A = \sqrt{\frac{m}{k}}\, v_{max} = \frac{v_{max}}{\omega_0}}$$

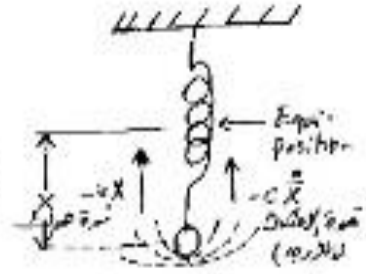
We also see from the energy equation that the maximum value of the speed, which we call v_{max}, occurs at $x = 0$. Accordingly, we can write

$$E = \frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2 \tag{3.3.7}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where $x = 0$ and $\dot{x} = \pm v_{max}$, and it is all potential energy at the extrema, where $\dot{x} = 0$ and $x = \pm A$.

3-3 Damped Harmonic Motion الحركة التوافقية المضمحلة

على شبيه حالة النابض في الموضوع السابق لو اعتبرنا ان هنالك وسط لزج يحيط بالنابض فان حركة النابض لن تكون محددة بالقوة المؤثرة (القوة المعيدة) التي تكون هي الوحيدة في حساب معادلة الحركة. بل يجب اضافة القوة الاخرى الموجودة وهي (القوة المؤخرة) (Retarding Force)



The damped harmonic oscillator

$$F = -kx \quad \text{--- (1)} \qquad F = -c\dot{x} \quad \text{--- (2)}$$

where: c - is const. of resistance

vescous retarding force (varying linearly with the speed)

∴ Equation of Motion then:

$$-kx - c\dot{x} = m\ddot{x} \quad \text{--- (3)}$$

or $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$ --- (4) The diff. eq. of Motion for the damped harmonic oscillator

وحل معادلة الحركة اعلاه نستخدم طريقة التجربة (trial sol.) باستخدام دالة exp. فنفرض ان الازاحة هي:

$$\boxed{x = Ae^{qt}} \quad \text{--- (5)}$$

$$\therefore \dot{x} = qAe^{qt}$$

$$\ddot{x} = q^2 Ae^{qt}$$

وبتعويض الدالة x ومشتقاتها في معادلة الحركة نحصل:

$$mq^2Ae^{qt} - cqAe^{qt} - kAe^{qt} = 0 \qquad \left[\div Ae^{qt}\right]$$

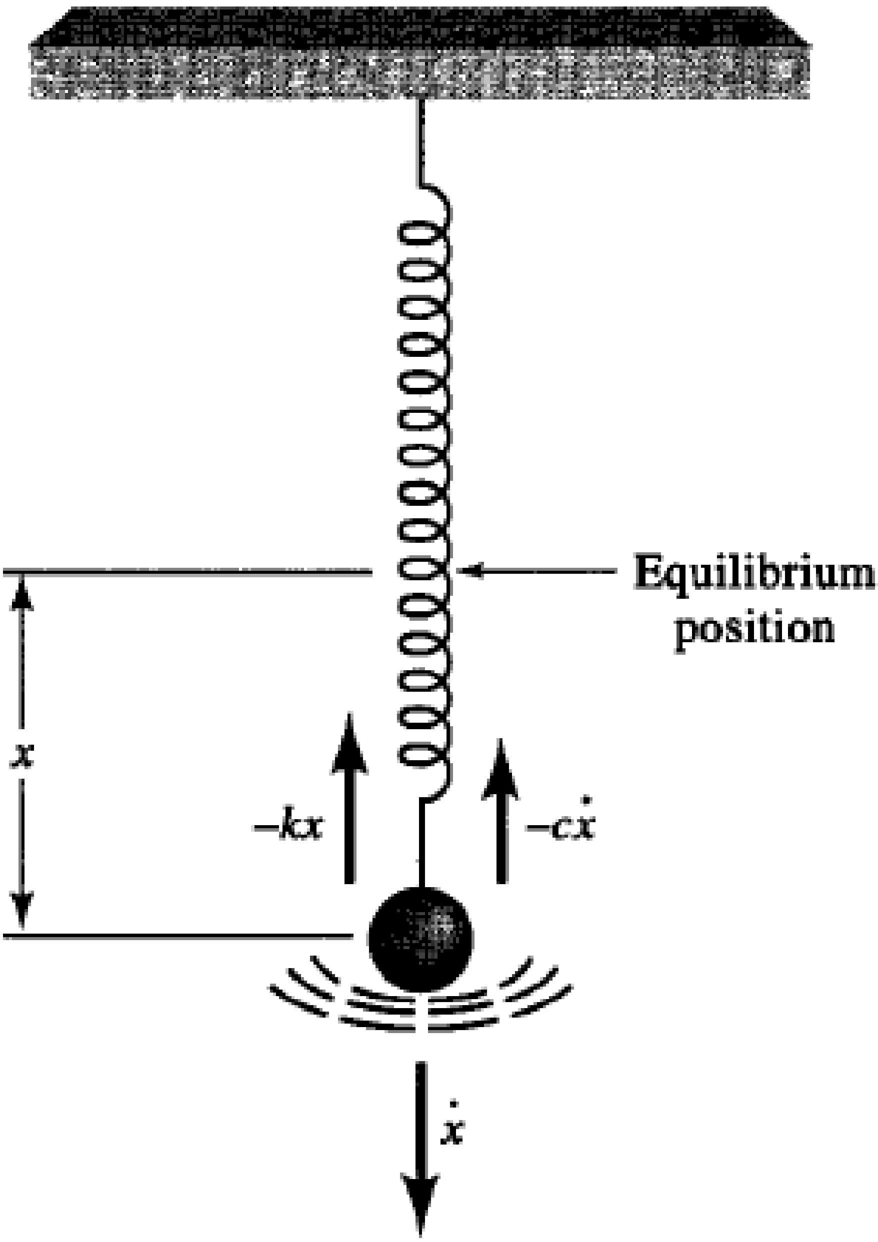
$$\boxed{mq^2 - cq - k = 0} \quad \text{--- (6)}$$

the roots are given by the well-known quadratic formula:

$$\boxed{q = \frac{-c \pm (c^2 - 4mk)^{1/2}}{2m}} \quad \text{--- (7)}$$

~~In case of ($c^2 > 4mk$) and $c^2 = 4mk$, q will be real and negative, (over damping) مفرط التخامد (critical damping) تخامد حرج~~

~~So the motion is nonoscillatory. (غير تذبذبية)~~


Equilibrium position
x
$-kx$
$-c\dot{x}$
$\dot{x}$

من معادلة المستقر، فإن (q) من الممكن ان تكتسب ثلاث انواع من القيم اعتمادا على ثابت المقاومة (C)، وهي كالاتي:

q - cases according to resistance constant (C):

Case I:

If $c^2 > 4mk$, then q will be real and negative and the motion will be nonoscillatory (overdamping)

$$q = -\begin{cases} \gamma_1 \\ \gamma_2 \end{cases} \longrightarrow x = \begin{cases} A_1 e^{-\gamma_1 t} \\ A_2 e^{-\gamma_2 t} \end{cases}$$

so the general solution for displacement is:

$$\boxed{x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}} \quad \text{--- } 8$$

Cas II:

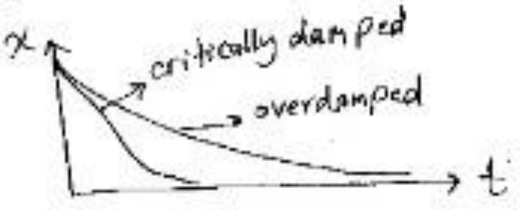
If $c^2 = 4mk$, the q will also be real and negative and the motion also nonoscillatory but (critical damped)

$$\boxed{q = \frac{-c}{2m}} \quad \text{--- } 9$$

$\therefore \gamma = \gamma_1 = \gamma_2$ & and we have: $-\gamma = q$

$\therefore \boxed{\gamma = -\left(\frac{-c}{2m}\right) = \frac{c}{2m}}$, sub. in eq. 8, we get:

$$x = A_1 e^{-\gamma t} + A_2 e^{-\gamma t} \boxed{= e^{-\gamma t}(A_1 + A_2)} \quad \text{--- } 9$$



coefficients. Let D be the differential operator d/dt. We "operate" on x with a quadratic function of D chosen in such a way that we generate Equation 3.4.4:

$$\left[D^2 + 2\gamma D + \omega_0^2\right]x = 0 \tag{3.4.5a}$$

We interpret this equation as an "operation" by the term in brackets on x. The operation by D^2 means first operate on x with D and then operate on the result of that operation with D again. This procedure yields $\ddot{x}$, the first term in Equation 3.4.4. The operator equation (Equation 3.4.5a) is, therefore, equivalent to the differential equation (Equation 3.4.4). The simplification that we get by writing the equation this way arises when we factor the operator term, using the binomial theorem, to obtain

$$\left[D + \gamma - \sqrt{\gamma^2 - \omega_0^2}\right]\left[D + \gamma + \sqrt{\gamma^2 - \omega_0^2}\right]x = 0 \tag{3.4.5b}$$

The operation in Equation 3.4.5b is identical to that in Equation 3.4.5a, but we have reduced the operation from second-order to a product of two first-order ones. Because the order of operation is arbitrary, the general solution is a sum of solutions obtained by setting the result of each first-order operation on x equal to zero. Thus, we obtain

$$x(t) = A_1 e^{-(\gamma - q)t} + A_2 e^{-(\gamma + q)t} \tag{3.4.6}$$

where

$$q = \sqrt{\gamma^2 - \omega_0^2} \tag{3.4.7}$$

The student can verify that this is a solution by direct substitution into Equation 3.4.4. A problem that we soon encounter, though, is that the above exponents may be real or complex, because the factor q could be imaginary. We see what this means in just a minute.

There are three possible scenarios:

I.	q real > 0	*Overdamping*
II.	q real $= 0$	*Critical damping*
III.	q imaginary	*Underdamping*

I. *Overdamped.* Both exponents in Equation 3.4.6 are real. The constants A_1 and A_2 are determined by the initial conditions. The motion is an exponential decay with two different decay constants, $(\gamma - q)$ and $(\gamma + q)$. A mass, given some initial displacement and released from rest, returns slowly to equilibrium, prevented from oscillating by the strong damping force. This situation is depicted in Figure 3.4.2.

II. *Critical damping.* Here $q = 0$. The two exponents in Equation 3.4.6 are each equal to γ. The two constants A_1 and A_2 are no longer independent. Their sum forms a single constant A. The solution degenerates to a single exponential decay function. A completely general solution requires two different functions and independent constants to satisfy the boundary conditions specified by an initial position and velocity. To find a solution with two independent constants, we return to Equation 3.4.5b:

$$(D + \gamma)(D + \gamma)x = 0 \tag{3.4.8a}$$

Case III

If $c^2 < 4mk$, then q will be complex, the real part of its ~~soluti~~ value gives an oscillatory motion, and the case is called (underdamped) تحت التخميد

$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c \mp \sqrt{c^2 \frac{4m^2}{4m^2} - 4mk \frac{4m^2}{4m^2}}}{2m}$$

$$= \frac{-c \mp \sqrt{4m^2\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m} = \frac{-c \mp 2m\sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}}{2m}$$

we have: $\gamma = \frac{c}{2m}$, $\omega_0 = \frac{k}{m}$

$$\therefore q = \frac{-c \mp 2m\sqrt{\gamma^2 - \omega_0^2}}{2m} = \frac{-c}{2m} \mp \frac{\cancel{2m}\sqrt{\gamma^2 - \omega_0^2}}{\cancel{2m}}$$

$$= -\gamma \mp \sqrt{\gamma^2 - \omega_0^2} = -\gamma \mp \sqrt{-(\omega_0^2 - \gamma^2)}$$

$$= -\gamma \mp \sqrt{-1}\sqrt{\omega_0^2 - \gamma^2} = -\gamma \mp i\sqrt{\omega_0^2 - \gamma^2}$$

$$\boxed{\therefore q = -\gamma \mp i\omega_1}$$

where: $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$

Also: $q_1 = -\gamma + i\omega_1$, $q_2 = -\gamma - i\omega_1$

Thus: $q = \begin{cases} -\gamma + i\omega_1 = q_1 \\ -\gamma - i\omega_1 = q_2 \end{cases}$

The displacement then:

$$x = A_1 e^{(-\gamma + i\omega_1)t} + A_2 e^{(-\gamma - i\omega_1)t}$$

$$\Rightarrow \boxed{x = e^{-\gamma t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]} \quad \text{--- (10)}$$

Upon using Uler formula: $e^{iu} = \cos u + i \sin u$:

$$\boxed{x = e^{-\gamma t}(a \sin \omega_1 t + b \cos \omega_1 t)}$$

where: $a = i(A_+ - A_-)$

$b = A_+ + A_-$

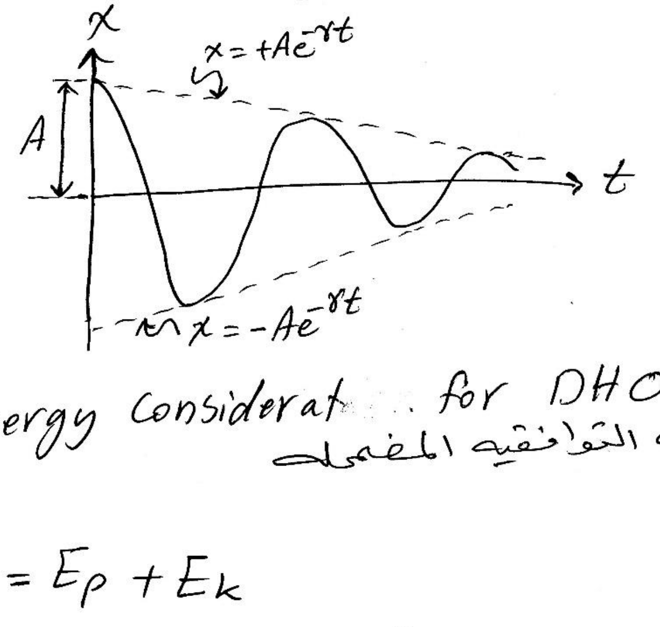
or:

$$\boxed{x = A e^{-\gamma t} \cos(\omega_1 t + \theta_0)}$$

where: $A = (a^2 + b^2)^{1/2}$

$\theta_0 = -\tan^{-1}(b/a)$

The displacement x is real, its amplitude $(Ae^{-\gamma t})$ decays exponentially with time.



3-4 Energy Considerat . for DHO:

اعتبارات الطاقة للحركة التوافقية المضمحلة

$$E_t = E_p + E_k$$

$$= \frac{1}{2} k x^2 + \frac{1}{2} m \dot{x}^2 \quad \text{--- (1)}$$

To find the time rate of change of (E), we have to defferentiate (E_t) w.r.t. time:

$$\frac{dE_t}{dt} = \frac{1}{\cancel{2}} k(\cancel{2} x \dot{x}) + \frac{1}{\cancel{2}} m(\cancel{2} \dot{x} \ddot{x})$$

$$=(k\bar{X}+m\ddot{\bar{X}})\dot{\bar{X}} \quad ---(2)$$

We have the eq. of motion for the damped harmonic oscillator

$$-k\bar{X}-c\dot{\bar{X}}=m\ddot{\bar{X}}$$

$$m\ddot{\bar{X}}+k\bar{X}=-c\dot{\bar{X}}$$

or

$$(k\bar{X}+m\ddot{\bar{X}})=-c\dot{\bar{X}} \quad ---(3)$$

- Substituting eq (3) within eq (2), we can write:

$$\frac{dE_t}{dt}=(-c\dot{\bar{X}})(\dot{\bar{X}}) \quad \boxed{=-c(\dot{\bar{X}})^2}$$

This eq. represents the rate at which energy (E_t) dissipated (تتبدد) into heat by friction الاحتكاك for DHO.

3-5 Forced Harmonic Motion. Resonance

الحركه التوافقيه المجبره. والرنين

في هذا الموضوع سنتناول DHM والتي تُقاد بواسطه قوة توافقيه خارجيه ($F_{ext.}$) اي ان القوة المسلطه تتغاير جيبيا (sinusouidally) مع الزمن.

$$F_{ext.}=F_0\cos(\omega t+\theta')=F_0 e^{i(\omega t+\theta')}$$

where: F_0 - amplitude
ω - angular frequancey

القوى المؤثرة على الجسم الان هي ثلاث :

$$F=-kX \quad ---(1)$$

$$F=-c\dot{X} \quad ---(2)$$

$$F=F_0 e^{i(\omega t+\theta')} \quad ---(3')$$

The diff. eq. of motion is therfore :

$$-k\bar{x} - c\dot{\bar{x}} + F_{ext.} = m\ddot{\bar{x}}$$

$$\boldsymbol{m\ddot{x} + c\dot{x} + kx = F_{ext} = F_o e^{i(wt+\theta)}}$$ Eq. of Motion for the Forced Harmonic Oscillator

معامل تخامد بوجود
قوة خارجية محركة

The suggested solution eq. is:

$$x = Ae^{i(wt+\acute{\theta})} \longrightarrow \dot{\bar{x}} = iwAe^{i(wt+\acute{\theta})} = iw\bar{x}$$

$$\ddot{\bar{x}} = i^2w^2Ae^{i(wt+\acute{\theta})} = i^2w^2\bar{x} = -w^2\bar{x}$$

بتعويض معادلة الحل ومشتقاتها (الاولى والثانية) في معادلة الحركة نحصل:

$$-mw^2Ae^{i(wt+\acute{\theta})} + icwAe^{i(wt+\acute{\theta})} + kAe^{i(wt+\acute{\theta})} = F_o e^{i(wt+\theta)}$$

نضرب المعادلة اعلاه بالدالة $e^{-i(wt+\acute{\theta})}$

$$-mw^2A + \underline{icwA} + kA = F_o(e^{iwt} \cdot e^{i\theta} \cdot e^{-iwt} \cdot e^{-i\acute{\theta}})$$

$$= F_o e^{i(\theta-\theta')}$$

$$= F_o[\cos(\theta-\acute{\theta}) + \underline{i\sin(\theta-\acute{\theta})}]$$

where: $\theta - \acute{\theta} = \varphi$, which is the angle of phase diff.

separation of real and imaginary terms, we get:

$$-mw^2A + kA = F_o\cos\varphi$$

$$A(k - mw^2) = F_o\cos\varphi \quad --- *$$

and

$$cwA = F_o\sin\varphi \quad --- ** \longrightarrow (i) \text{ is eliminated}$$

من الطرفين

deviding eq ~~**~~ on eq* we get:

$$\frac{cwA}{A(k-mw^2)} = \frac{F_o\sin\varphi}{F_o\cos\varphi}$$

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$$\therefore \tan\varphi = \frac{c\omega}{(k - m\omega^2)} = \frac{\frac{c}{m}\,\omega}{\frac{k}{m} - \frac{\cancel{m}\omega^2}{\cancel{m}}}$$

بعد قسمة البسط والمقام بـ (m)

We have: $\gamma = \frac{c}{2m}$ (damped frequarey)

$$\boxed{\varphi = \tan^{-1}\left(\frac{2\gamma\omega}{w_o^2 - w^2}\right)}$$

$$\therefore \boxed{2\gamma = \frac{c}{m}} \quad , \quad \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

$$\therefore \boxed{\tan\varphi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}}$$

The difference in phase between the applied driving force and the steady-state response

Solving for $\underline{A}$ (the amplitude of the steady-state oscillation), yields: (squaring both eq (*) & (**), then adding)

اعادة ترتيب المعادلتين بعد تربيعهما للتخلص من sin و cos

$$\therefore A^2(k - m\omega^2)^2 = F_0^2 \cos^2\varphi$$

$$c^2\omega^2A^2 = F_0^2 \sin^2\varphi$$

$$A^2(k - m\omega^2) + c^2\omega^2A^2 = F_0^2(\cos^2\varphi + \sin^2\varphi)$$

$$= F_0^2$$

$$A^2[(k - m\omega^2)^2 + c^2\omega^2] = F_0^2$$

(بقسمة البسط والمقام m)

$$\therefore A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 - c^2\omega^2}} = \frac{\frac{F_0}{m}}{\sqrt{\left(\frac{k}{m} - \frac{m\omega^2}{m}\right)^2 + \frac{c^2}{m^2}\omega^2}}$$

$$\boxed{A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}} \quad \text{---}(\#)$$

سعة التذبذب عند الحالة المستقرة

steady state Amplitude oscillation

The equation that relating the amplitude (A) to the Impressed driving frequency (ω).

(59) في معادلة السعة، فان العظم قيمة للسعة (A) تتحقق فقط عندما تصبح $(\omega = \omega_r)$ اي عند تردد الرنين Resonance frequency ولايجاد قيمة تردد الرنين يجب ان نشتق معادلة السعة بالنسبة ~~للرنين~~ للتردد ونساوي الناتج بالصفر.

$$\frac{dA}{d\omega} = \frac{d}{d\omega}\left[\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}\right] = 0$$

$$= \frac{F_0}{m}\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2)\right]^{-\frac{1}{2}} = 0$$

$$= \frac{F_0}{m}\frac{d}{d\omega}\left[\omega_0^4-2\omega_0^2\omega^2+\omega^4+4\gamma^2\omega^2\right]^{-\frac{1}{2}} = 0$$

$$= \frac{F_0}{m}\left[-\frac{1}{2}\right]\left[\omega_0^4-2\omega_0^2\omega^2+\omega^4+4\gamma^2\omega^2\right]^{-\frac{1}{2}-1}$$
$$\left[\text{zero}-(2)(2)\omega_0^2\omega+4\omega^3+(4)(2)\gamma^2\omega\right] = 0$$

$$= \frac{-F_0}{2m}\frac{4\omega\omega_0^2+4\omega^3+8\gamma^2\omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{3/2}} = 0$$

$$= \frac{-F_0}{\cancel{2}m}\frac{{}^{2}\cancel{4}\omega\left[-\omega_0^2+\omega^2+2\gamma^2\right]}{\left[(\omega_0^2-\omega^2)+4\gamma^2\omega^2\right]^{3/2}} = 0 \rightarrow \boxed{2\omega = ?}$$

$$\left[\frac{(-F_0)(2\omega)}{m}\right]$$

$$= \frac{-F_0}{m}\frac{2\omega\left[\omega^2-\omega_0^2+2\gamma^2\right]}{\left[(\omega_0^2-\omega^2)+4\gamma^2\omega^2\right]^{3/2}} = 0$$

$$\frac{\omega^2-\omega_0^2+2\gamma^2}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{3/2}} = 0$$

بما ان الكسر = صفر، فان المقام لا يمكن ان يساوي صفر وذلك يعني ان (البسط = صفر)

$$\omega^2 - \omega_0^2 + 2\gamma^2 = 0$$

$$\rightarrow \omega^2 = \omega_0^2 - 2\gamma^2$$

for $\boxed{\omega = \omega_r = (\omega_0^2 - 2\gamma^2)^{1/2}}$ Resonance frequency ---(# #) (frequency for max. amplitude)

In case of weak damping, that is, $(c << 2\sqrt{mk})$ or $(\gamma << \omega_0)$ then: $\boxed{\omega_r \simeq \omega_0}$

اي سيكون التردد الرنيني مقارباً جداً للتردد
المتذبذب الحر (بدون اضمحلال).

$$\omega_r = (\omega_0^2 - 2\gamma^2)^{1/2}$$

Binomial Theorem for expansion:

$\cancel{(\omega_0^2 - 2\gamma^2)^{1/2}} \cancel{=}$ $\boxed{(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \cdots}$

$$(\omega_0^2 - 2\gamma^2)^{1/2} =$$

* The steady-state amplitude at the resonant frequency which we shall call ($A_{max.}$) is obtained as the following:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}$$ ---(#) (amplitude at steady-state condition)

eq(# #): $\underbrace{\omega^2 = \omega_0^2 - 2\gamma^2}_{---①} \longrightarrow \underbrace{2\gamma^2 = \omega_0^2 - \omega^2}_{---②}$

sub. eqs ① + ② in eq(#) yields:

$$A_{max.} = \frac{F_0/m}{\sqrt{(2\gamma^2)^2 + 4\gamma^2(\omega_0^2 - 2\gamma^2)}} = \frac{F_0/m}{\sqrt{4\gamma^4 + 4\gamma^2\omega_0^2 - 8\gamma^4}}$$

$$A_{max} = \frac{F_0/m}{\sqrt{4\gamma^2 W_0^2 - 4\gamma^4}} = \frac{F_0/m}{\sqrt{4\gamma^2(W_0^2-\gamma^2)}}$$

$$\Rightarrow A_{max.} = \frac{F_0/m}{2\gamma\sqrt{(W_0^2-\gamma^2}} \longrightarrow A$$ العظمى (الاحد)

For Weak daming : $\gamma \ll W_0$

$$A_{max} = \frac{F_0}{2\gamma m W_0}$$

$$\gamma = \frac{c}{2m} \rightarrow c = 2\gamma m$$

$$\therefore \boxed{A_{max.} = \frac{F_0}{cW_0}}$$ steady-state amplitude at the resonant frequency.

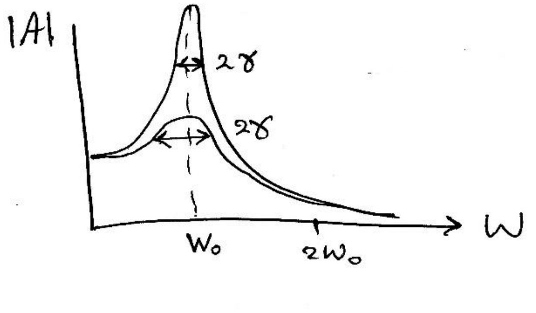
$$A = \frac{A_{max.}\,\gamma}{\sqrt{(W_0-W)^2+\gamma^2}}$$ (relation between amplitude and max. Amplitude) at weak damping.

This equation shows that, when $\boxed{|W_0 - W| = \gamma}$, then:

$$A = \frac{A_{max.}\,\gamma}{\sqrt{\gamma^2+\gamma^2}} = \frac{A_{max.}\,\cancel{\gamma}}{\cancel{\gamma}\sqrt{2}}$$

$$\Rightarrow \boxed{A^2 = \frac{1}{2} A^2_{max.}}$$

* من المعادلة $|W_0 - W| = \gamma$
يتضح بأن (γ) تمثل العرض مقياس
لمنحنى الرنين.



Another way of designating the sharpness of the resonance peak is in terms of parameter (Q) called the quality factor of a resonant system. It is defined as:

$$Q = \frac{\omega_r}{2\gamma}$$

or, for weak damping: $\boxed{Q \simeq \frac{\omega_0}{2\gamma}}$

Thus the width ($\Delta\omega$) at the half-energy points is approximately: $\boxed{\Delta\omega = 2\gamma \simeq \frac{\omega_0}{Q}}$

$\because \omega = 2\pi f \longrightarrow \Delta\omega = \Delta 2\pi f = 2\pi\Delta f$

also: $\omega_0 = 2\pi f_0$

$$\therefore \frac{\Delta\omega}{\omega_0} = \frac{\cancel{2\pi}\Delta f}{\cancel{2\pi} f_0} = \boxed{\frac{\Delta f}{f_0} \simeq \frac{1}{Q}}$$

eg. (1 P. 71) (H.W.)

① A particle of mass (m) is attached to a spring of stiffness k. The damping is such that ($\gamma = \omega_0/4$). Find the ~~natural~~ resonance frequency, and the damped oscillator, and quality factor

Solution

$$\omega_r = (\omega_0^2 - 2\gamma^2)^{1/2} = \left(\omega_0^2 - \frac{2\omega_0^2}{16}\right)^{1/2}$$

$$= \omega_0\sqrt{\frac{7}{8}} = \sqrt{\frac{k}{m}}\sqrt{\frac{7}{m}}$$

for the resonance frequency in angular measure. The quality factor is given by:

$$Q = \frac{\omega_r}{2\gamma} = \frac{\omega_0\left(\frac{7}{8}\right)^{1/2}}{2(\omega_0/4)} = 2\sqrt{\frac{7}{8}} = 1.87$$

② If the applied frequency is ($\omega_0/2$) for the above oscillator, find the phase angle φ!

solution

$$\tan\varphi = \frac{2\gamma\omega}{(\omega_0^2 - \omega^2)} = \frac{2\left(\frac{\omega_0}{4}\right)\left(\frac{\omega_0}{4}\right)}{\omega_0^2 - \frac{\omega_0^2}{4}} = \frac{\frac{1}{4}\omega_0^2}{\omega_0^2\left(1 - \frac{1}{4}\right)}$$

$$\tan \varphi = \frac{1/4}{3/4} = \frac{1}{3}$$

$$\longrightarrow \varphi = \tan^{-1}\left(\frac{1}{3}\right) = 18.5^\circ$$