Rectilinear Motion of a Particle

2.1 Newton's Laws of Motion:

I. Every body continues in its state of rest, or of uniform motion in a stright line, unless it is compelled to charge that state by forces Impressed upon it.

II. The charge of motion is proporitional to the motive force Impressed and is made in the direction of the line in which that force is Impressed.

III To every action there is always Imposed an equal reaction, or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

2.2 Mass and Force: Newton's Second and Third Laws

Consider two masses, m_1 & m_2 attached by a spring and initially at rest. Imagine somone pushing the two masses together, compressing the spring and then suddenly releasing them so that they fly apart, attaining speed v_1 & v_2.

We define the ratio of the two masses to be:

$$\frac{m_2}{m_1} = \left|\frac{v_1}{v_2}\right| \quad --- (1)$$

فاذا ما ضربنا الوسطين في الطرفين للمعادلة (1) فان الناتج هو linear moment الزخم الخطي . اي

$$m_2 v_2 = m_1 v_1 \quad --- (\#)$$

حيث يتساوى الزخمان بفعل كون الكتلتين مترابطتين بالنابض وبالاستفادة من قانون حفظ الزخم الخطي ، فان مجموع التغير في الزخم ~~الكتله الاولى يجب ان يساوي~~ الخطي يجب ان يساوي صفر.

وبذلك نضع المعادلة الاتي :

$$\boxed{\Delta m_2 v_2 = -\Delta m_1 v_1} \quad --- (2)$$

حيث ان $(v_1)_0 = (v_2)_0 = 0$ لكون الكتلتين كانتا في حالة سكون في البدء، وتشير المعادلة (2) كذلك الى ان الكتلتين تحركتا باتجاهين متعاكسين.

وباجراء $\left(\lim_{\Delta t\to 0}\frac{1}{\Delta t}\right)$ للمعادلة (2) نحصل

$$\frac{d}{dt}(m_1 v_1) = -\frac{d}{dt}(m_2 v_2) \quad --- (*)$$

ثم

$$\lim_{\Delta t\to 0}\frac{\Delta m_2 v_2}{\Delta t} = -\lim_{\Delta t\to 0}\frac{\Delta m_1 v_1}{\Delta t}$$

حيث ان :

$$\frac{d}{dt}(mv)$$

تمثل التغير في الزخم الخطي charge of linear momentum

So the Second law can be rephrased as follows,

"The time rate of change of an object's linear momentum is proportional to the Impressed force."

ماذا ما علمنا ان التغير الزمني للزخم يتناسب مع القوة المسلطة على الكتلة :

$$\vec{F} \propto \frac{d}{dt}(m\vec{v})$$

$$= k \frac{d}{dt}(m\vec{v})$$

$$= k\, m \frac{dv}{dt}$$

$$= k m a$$

where a is the resultant acc. of a mass (m) subjected to a force $\vec{F}$.

k is the proportionality constant.

- let : k = 1

$$\therefore \boxed{\vec{F} = m\vec{a}} \quad --- (3)$$

ان القوة $\vec{F}$ في يسار المعادلة ③ تمثل صافي القوة المؤثرة على الكتلة (m) (net force) والتي هي محصلة القوى المؤثرة على هذه الكتلة.

وبتعويض هذه النتيجة في المعادلة (*) نحصل:

$$\boxed{\vec{F}_1 = -\vec{F}_2} \quad --- (4)$$

وهو قانون نيوتن الثالث.

2-3 Linear Momentum الزخم الخطي

$$\vec{P} = m\vec{v} \quad \text{---} (1)$$

$$\therefore \quad \vec{F} = \frac{d\vec{P}}{dt} \quad \text{--} (2) \quad \text{Newton's 2nd law}$$

بدلالة التغاير الزمني للزخم الخطي

sub. eq (2) in eq $(\#)$, we get:

$$\frac{d}{dt}(\vec{P}_1 - \vec{P}_2) = 0$$

وهي معادلة تصف كتلتان في حالة تفاعل متبادل ، فاذا كاملنا بالنسبة للزمن نحصل على :

$$\vec{P}_1 + \vec{P}_2 = \text{Const.}$$

أي أنه استنادا الى قانون نيوتن الثالث فان محصلة الزخم لجسمين في حالة تفاعل يكون ثابت .

وهي حالة خاصة بالنسبة للحالة الاكثر عمومية وهي حالة ان الزخم الخطي لنظام معزول ميكانيكيه محفوظة Conserved

محصلة

Example 2.1.2

A space ship of mass (M) is traviling in deep space with velocity $v_i = 20\,km/s$ relative to the sun. It ejects a rear stage of mass (0.2M) with relative speed ($u = 5\,km/s$). What then is the velocity of the spaceship?

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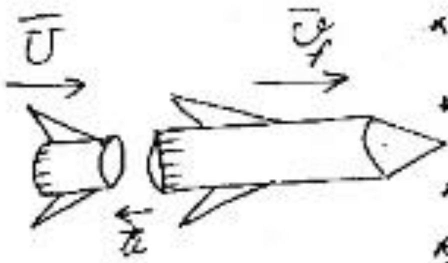
Sol.

neglecting the gravitational force of the sun,

عند اعتبار قوة جاذبية الشمس على المركبة (صفر) وباعتبار اننا نتعامل مع نظام مغلق (المركبة وجزءها الخلفي) اي كنظام واحد، عليه فان الزخم الخطي الكلي يجب ان يكون محفوظ اي:



* mass of spaceship = M
* mass of ejected stage = 0.2M
* $\bar{v}_i$ = 20 km/s
* $\bar{u}$ = 5 km/s

$\boxed{\bar{v}_f = ?}$

$$\Delta \bar{P} = \bar{P}_f - \bar{P}_i = 0$$

$$\therefore \bar{P}_f = \bar{P}_i \quad --- (1)$$

where: $\bar{P}_f$: final linear momentum.
$\bar{P}_i$: initial linear momentum.

So,

$$\bar{P}_i = M \bar{v}_i \quad --- (2)$$

$$\bar{P}_f = (0.2M\bar{U}) + (0.8M \bar{v}_f) \quad --- (3)$$

where:

$$\bar{U} = \bar{v}_f - \bar{u} \quad --- (4)$$

سرعة الجزء الخلفي المقذوف

where $\bar{u}$ is the velocity of ejected rear part with respect to the space ship.

sub. eq (4) in eq (3) yields, then equating the result with eq (2)

$$\bar{P}_f = 0.2M(\bar{v}_f - \bar{u}) + 0.8M \bar{v}_f$$

$$\therefore 0.2M(\bar{v}_f - \bar{u}) + 0.8M\bar{v}_f = M\bar{v}_i \qquad \div M$$

$$0.2\bar{v}_f - 0.2\bar{u} + 0.8\bar{v}_f = \bar{v}_i$$

$$\bar{v}_f = \bar{v}_i + 0.2\bar{u} = 20 +$$

2-4 Rectilinear Motion: Uniform Acceleration Under aconstant force

يقال عن الحركة بأنها (rectilinear) عندما ينتقل الجسم المتحرك محافظاً على مساره بشكل خط مستقيم . عليه يمكن كتابة معادلة الحركة بالشكل الآتي .

$$\bar{F}_x(x,\dot{x},t) = m\ddot{x} = ma$$

على فرض ان الجسم يتحرك على محور x فاننا سنستخدم الرمز $\bar{F}$ بدلا $\bar{F}_x$

$v \to \dot{x}$, $a \to \ddot{x}$

- عندما تكون محصلة القوة $\bar{F}$ المؤثرة على الجسم ثابتة عندئذ يكون التعجيل ايضا ثابت اي:

when $\bar{F} = \text{const.}$ then $\bar{a} = \text{const.}$

$$\frac{d\bar{v}}{dt} = \frac{\bar{F}}{m} = \bar{a} = \text{const}$$

$$\therefore \int_{v_0}^{v} d\bar{v} = \int_0^t \bar{a}\,dt$$

$$v - v_0 = at \longrightarrow \boxed{\bar{v} = at + v_0} \quad \text{---} \; ①$$

$$\frac{d\bar{x}}{dt} = at + v_0$$

$$\int_{x_0}^{x} dx = \int_0^t at\,dt + \int_0^t v_0\,dt$$

$$\boxed{x - x_0 = \frac{1}{2}at^2 + v_0 t} \quad \text{---} \; ②$$

From eq①:

$$\bar{v} - \bar{v}_0 = at \longrightarrow \boxed{t = \frac{\bar{v} - \bar{v}_0}{a}} \quad \text{---} \; ③$$

sub. eq③ in eq② we get:

$$x - x_0 = \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 + v_0\left(\frac{v - v_0}{a}\right) \qquad * \; 2a$$

$$2a(x-x_0) = \cancel{a^2}\left(\frac{v-v_0}{\cancel{a}}\right)^2 + 2\cancel{a}\left(\frac{v-v_0}{\cancel{a}}\right)v_0$$

$$2a(x-x_0) = v^2 \cancel{-2vv_0} + v_0^2 \cancel{+2vv_0} - 2v_0^2$$

$$\boxed{2a(x-x_0) = v^2 - v_0^2} \quad ---(4)$$

حيث تصف المعادلات 1 / 2 / 3 / 4 جسم يتحرك بتعجيل متجانس

where:
v_0 is the velocity and x_0 is the position at $(t=0)$.

هناك عدة امثلة على التعجيل المتجانس منها السقوط الحر للاجسام (عند اهمال قوة الاحتكاك بالهواء) حيث يصبح التعجيل $(a = g = 9.8\,m/s^2)$ - ويعبر عن القوة المؤثرة على الجسم (قوة جذب الارض) بقوة الجاذبية وتتمثل بوزن الجسم (mg)

$$\bar{F} = mg = \text{weight}$$

Example (2.2.1)

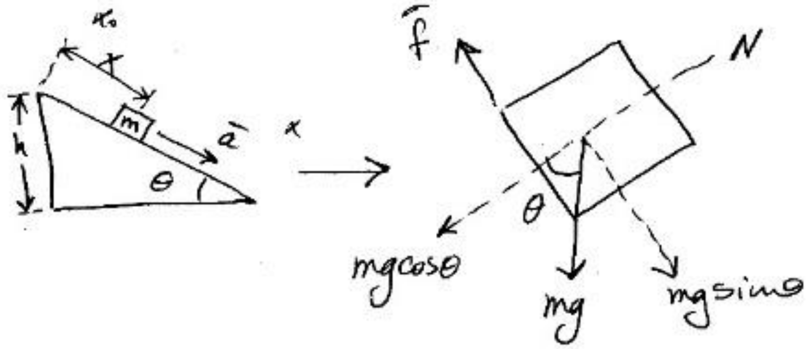
consider a block that is free to slide down a smooth, frictionless plane that is Inclined at an angle (θ) to the horizontal, as shown in figure. If the height of the plane is (h) and the block is released from rest at the top, what will be its speed when it reaches the bottom?

Sol.

Angle of Inclination : θ

hight of slide : h

mass of block : m



من تحليل مركبات القوى المؤثرة على الجسم نحصل على الشكل اعلاه
ومنه تتواجد احتماليتان:

① في حالة كون قوة الاحتكاك مهملة $(\bar{f}=0)$

$$\bar{F} = m\bar{a} \quad ---①$$

$$\bar{a} = \frac{\bar{F}}{m}$$

For x-axis component,

$$\bar{F} = mg\sin\theta \quad ---②$$

equating equations ① & ②

$$\cancel{m}\bar{a} = \cancel{m}g\sin\theta \longrightarrow a = g\sin\theta \quad ---③$$

From the fig. $\sin\theta = \frac{h}{x-x_0} \longrightarrow x-x_0 = \frac{h}{\sin\theta} \quad ---④$

using the equation of motion:

$$2a(x-x_0) = v^2 - v_0^2 \quad , \text{where } v_0 = 0$$

$$v^2 = 2a(x-x_0)$$
$$= 2g\sin\theta\left(\frac{h}{\sin\theta}\right)$$
$$= 2gh$$

$$\therefore v = \sqrt{2gh}$$

$\bar{f} \propto N$ ② عند كون قوة الاحتكاك $\bar{f}$ غير مهملة ،

$$\bar{f} = \mu_k N \quad \text{---} (5)$$

where: N: is the normal force

μ_k: coeff. of kinitic friction

From the fig.

$$N = mg\cos\theta \quad \text{---} (6)$$

sub. eq (6) in eq(5), we get:

$$\bar{f} = \mu_k \, mg \cos\theta \quad \text{---} (7)$$

$\boxed{mg\sin\theta > \mu_k mg\cos\theta}$

the net force acting on the block in x-direction is:

ويكون الجسم يتحرك نحو الاسفل (لا اعلى في هذه الحالة) لان اتجاه قوة الاحتكاك نحو الاعلى او ان

$$\bar{F} = mg\sin\theta - \mu_k mg\cos\theta = m\vec{a} \quad (\div m)$$

$$a = g(\sin\theta - \mu_k \cos\theta)$$

* تراعى الإشارة عندما يكون اتجاه حركة الجسم نحو الاعلى فيصبح التعجيل تباطؤ

2.5 Forces that Depend on Position: The concepts of Kinetic and Potential Energy.

يعتمد

It is often true that the force a particle experiences depends on the particle's position with respect to other bodies. for example $F_o = K \frac{q_1 q_2}{r^2}\hat{r}$ & $\bar{F}_G = G\frac{m_1 m_2}{r^2}$ where r is the distance between q_1 and q_2 or between m_1 & m_2. It is also applieds to forces of elastic tension or compression. Then the differential equation for rectilinear motion is simply:

$$F(x) = m\ddot{x} \quad \text{---} (1)$$

(القوة غير معتمدة على السرعة او الزمن)

where: $\ddot{x} = \frac{d\dot{x}}{dt}$, application of chain rule to write the acceleration in the following way:

$$\frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

So, eq(1) may be written:

$$F(x) = m v \frac{dv}{dx}$$

$\frac{2}{2}$ نضرب

$$= \frac{m}{2}\frac{2v\,dv}{dx} = \frac{1}{2}m\frac{dv^2}{dx}$$

$\because \frac{1}{2}m$ is a constant, so we can write

$$F(x) = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = \frac{dT}{dx}$$

where $T = \frac{1}{2}mv^2$ is the kinetic energy of the particle.

بضرب الطرفين في dx:

by Integration:

$$F(x)\,dx = dT$$

$$\int_{x_0}^{x} F(x)\,dx = \int_{T_0}^{T} dT$$

The integral $\left(\int F(x)\,dx\right)$ is the work done on the particle by the impressed force $F(x)$. (T) is the final kinetic energy, (T_0) is the initial kinetic energy

من القوس

$$\therefore \boxed{W = \int F(x)\,dX = T - T_0}$$ work and kinitic energy equation (28)

Let us define a function $V(x)$ such that:

$$-\frac{dV(x)}{dX} = F(X)$$

where $V(X)$ is called the potential energy

$$\therefore \int_{X_0}^{X} F(x)\,dX = -\int_{V_0}^{V} dV(X)$$

$$\therefore W = -[\,V(X) - V_0(X)\,]$$

$$\boxed{= -V(X) + V_0(X) = T - T_0}$$ Work and potential energy Equation

or:

$$\boxed{T_0 + V_0(x) = V^{(x)} + T = \text{Constant} = E}$$ Energy Equation

where: E is the total Energy

$$\therefore E = \frac{1}{2} m v_0^2 + V_0(X) \quad \text{---} \quad \text{الحالة الابتدائية}$$

$$= \frac{1}{2} m v^2 + V(X) \quad \text{----} \quad \text{الحالة النهائية}$$

This is the case in which the Impressed force is a function only of the Position (X), such a force is said to be <u>conservative</u>

من معادلة الطاقة الكلية يمكن اشتقاق الآتي:

$$\frac{1}{2}mv^2 + V(x) = E$$

$$\frac{1}{2}mv^2 = E - V(x) \qquad * \frac{2}{m}$$

$$v^2 = \frac{2}{m}[E - V(x)]$$

$$\boxed{v = \mp\sqrt{\frac{2}{m}[E - V(x)]}} \quad \text{-----} *$$

Equation of velocity as a function of x

$$\because \quad v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \mp\sqrt{\frac{2}{m}[E - V(x)]}$$

$$\int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{dx}{\mp\sqrt{\frac{2}{m}[E - V(x)]}}$$

$$\boxed{t - t_0 = \int \frac{\mp\, dx}{\sqrt{\frac{2}{m}[E - V(x)]}}}$$

Equation of (t) as a function of (x).

للمعادلة (*) يتضح ان السرعة v تكتسب قيم حقيقية real فقط عندما يتحقق الشرط $(V(x) \leqslant E)$ للمتغير x (متغيرا الموقع) وهذا يعني فيزيائيا ان الجسم مقيد في حركته في هذه المنطقة

For the velocity Equation:

$$\bar{v} = \mp\sqrt{\frac{2}{m}[E - V(x)]}$$

① If $E > V(x)$

the velocity has a real value

② If $E = V(x)$

the velocity equals to zero

③ If $E < V(x)$

the velocity has an imagenary value. (not allowed)

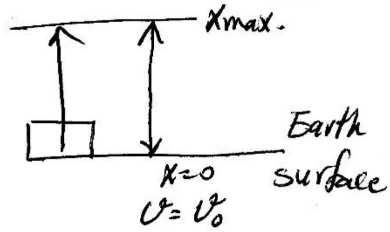
H.W Given the force equation $\bar{F} = \frac{k}{x}$, where k is a constant. Find the velocity as a function of x.

Example Free Fall

A body is projected upward (in the positive x-direction) with initial speed (v_0), choosing ($x=0$) as an initial of projection. Find the maximum height attained ($\ddot{x}$) by the body and then find the equation of time (t) in terms of (g).

sol- v_0, initial position: x_0, $x_{max.} = ?$, $t = ?$

gravity ($F = mg$) The equation for the force of gravity where: g - is the acc. due to gravity.
(g) is taken to be (+ve) for Falling bodies. while it taken to be (-ve) incase of the body goes away from the earth.



$F = -mg$ --- (1)

we have the relation

$\int F(x)\,dx = -V(x) + C$ --- (2)

(work - Potential energy)

sub. eq (1) in eq (2)

$\int -mg\,dx = -V(x) + C$

$-mgx = -V(x) + C$

$\therefore$ Potential energy $V(x) = mgx$ ---- (3)

$\because E = \frac{1}{2}mv^2 + V(x)$

$= \frac{1}{2}mv^2 + mgx$

عند مستوى الانطلاق فان $x=0$ ، $v=v_0$ ، وعليه تصبح الطاقة

$E = \frac{1}{2}mv_0^2 + mg(0)$

$\therefore = \frac{1}{2}mv_0^2$

ومن اعتبار ان القوة المؤثرة على الجسم المقذوف هي قوة محافظة
(اي لا وجود لتأثير مقاومة الهواء ((الاحتكاك)) فانه يمكن تطبيق معادلة
الطاقة الكلية الاتية

$E_i = \text{const.} = E_f$

$0 + \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx$ $\quad * \frac{2}{m}$

$v_0^2 = v^2 + 2gx$

$\therefore x = \frac{v_0^2 - v^2}{2g}$

at max. altitude:

$v = 0$, $X = X_{max}$

$$\therefore X_{max} = \frac{v_0^2 - 0}{2g}$$

Using the following equation of motion: $\bar{v} = at + \bar{v}_0$

which will be in the form:

$\bar{v} = gt + \bar{v}_0$ For this problem: $\bar{v} = -gt + \bar{v}_0$

at $X_{max.}$, $\bar{v} = 0$, thus:

$$0 = -gt + \bar{v}_0$$

$$\therefore gt = \bar{v}_0$$

$$\rightarrow \boxed{t = \frac{\bar{v}_0}{g}}$$

2-6 Variation of Gravity with height

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تعاد الصياغة كليا

Newton's Law of gravity: $F = mg$

where g is constant $= 9.8$ (Nt/kg) m/s^2

في الحقيقة فان هذه العلاقة تكون صحيحة فقط عند سطح الارض، وذلك تكون قوة الجاذبية هي احدى قوانين التربيع العكسي وتعتبر دالة لكل من المربع (r) بين مركزي اي كتلتين (مثل مركز الارض وجسم مقذوف) وكذلك لمقدار كتلة اي جسم وكتلة الارض. عندئذ يكون قانون الجذب العام كالاتي:

Newton's Law of Gravity

$$\boxed{F_r = -G \frac{Mm}{r^2}} \quad \text{--- (1)}$$

G – Newoton's constant of gravitation

عندما يكون الجسم قيد الدراسة على سطح الارض فان القوتين اعلاه تتساويان اي يمكن كتابة الاتي:

عند سطح الارض فقط

$$mg = G \frac{Mm}{r_e^2} \longrightarrow \boxed{g = G \frac{M}{r_e^2}}$$

acc. of gravity at Earth's surface

تعجيل ثابت

(2-6) Variation of Gravity with Height:

We assumed that g was constant. Actually, the force of gravity between two particles is inversely proportional to the square of the distance between them (Newton's law of gravity). Thus, the gravitational force that the Earth exerts on a body of mass m is given by:

لقد افترضنا سابقاً بان g هو ثابت (يبلغ مقداره $9.8\ m/S^2$) لكن في الحقيقة فأن قوة الجاذبية بين جسمين تتناسب عكسيا مع مربع المسافة بينهما (وهو قانون نيوتن للجاذبية). عليه فأن قوة الجاذبية التي تبديها الارض بالنسبة لجسم كتلته m تعطى بالمعادلة الاتية:

$$\boxed{\vec{F}_r = -\frac{GMm}{r^2} \quad ---(1)}$$

$$[\textit{Newton's law of Gravity between the earth and paticle of mass m}]$$

in which G is Newton's constant of gravitation, M is the mass of the Earth, and r is the distance from the center of the Earth to the body.

We know that there is a relation between the force and potential energy,

$$\vec{F} = -\frac{\partial V}{\partial r} \quad \rightarrow \partial V = \vec{F}\partial r \rightarrow dV = \vec{F}dr \rightarrow \int dV = \int \vec{F}dr$$

Substituting eq. (1) into last eq. we get:

$$\int dV = \int -\frac{GMm}{r^2}dr$$

$$V(r) = -GMm\int r^{-2}\,dr = -GMm\left(\frac{r^{-2+1}}{-1}\right)$$

$$\boxed{\therefore V(r) = \frac{GMm}{r} \quad ---(2)}$$

$$[\textit{Invers First} - \textit{Power Potantial Energy Function}]$$

ولجسم يتحرك خطياً عند مستوى سطح الارض، فأن حركته تتأثر بقوة وحيدة هي قوة الجاذبية الارضية (وزن الجسم)، (على فرض ان قوة الجاذبية قوة محافظة) وحركته يحكمها قانون نيوتن الثاني (وفيها يكتب التعجيل a بالرمز g)، اي:

$$\vec{F} = mg = mr$$

Thus the ***equation of motion*** for the particle m is written as the following:

$$\boxed{mg = m\ddot{r} = -\frac{GMm}{r^2} \quad ---(*)}$$

But if we apply the chain rule for the left hand side:

$$\ddot{r} = \frac{d\dot{r}}{dt}\frac{dr}{dr} = \frac{dr}{dt}\frac{d\dot{r}}{dr} = \dot{r}\frac{d\dot{r}}{dr}$$

$$\therefore \quad m\dot{r}\frac{d\dot{r}}{dr} = -\frac{GMm}{r^2}$$

Integrating both sides of the last equation with respect to $\dot{r}$ and r, getting:

$$m\int \dot{r}\,d\dot{r} = -GMm\int r^{-2}\,dr$$

$$\frac{1}{2}m\dot{r}^2 = \frac{GMm}{r}$$

$$\boxed{\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = E = E_t \quad ----(3)}$$

[*Free Falling Energy Equation*]

In which E is the constant of integration. This is in fact the energy equation: the sum of kinetic and potential energy remains constant throughout the motion of a falling body.

لنطبق الان معادلة الطاقة اعلاه على قذيفة (Projectile) تنطلق نحو الاعلى من على سطح الارض وبانطلاق ابتدائي مقداره ϑ_o ، عليه فإن الثابت E يحسب الان وفقا للشروط الابتدائية (Initial Conditions):

$$\frac{1}{2}m{\vartheta_o}^2 - \frac{GMm}{r_e} = E \quad ------(4)$$

Where r_e is the radius of the earth. Now, in order to find the speed of the projectile at any height x above the earth's surface, combining the last two energy equations, we can get:

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = \frac{1}{2}m{\vartheta_o}^2 - \frac{GMm}{r_e}$$

$$\frac{1}{2}m{\vartheta_o}^2 - \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r} - \frac{GMm}{r_e}$$

$$\frac{1}{2}m\left({\vartheta_o}^2 - \dot{r}^2\right) + GMm\left(\frac{1}{r} - \frac{1}{r_e}\right)$$

Substituting by ($\vartheta = \dot{r}$), then multiply the equation by the factor $\frac{2}{m}$, we get;

$$\vartheta_o{}^2 - \vartheta^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right) = 0$$

$$\vartheta^2 = \vartheta_o{}^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right)$$

But: $r = r_e + x$

$$\therefore \vartheta^2 = \vartheta_o{}^2 + 2GM\left(\frac{1}{r_e + x} - \frac{1}{r_e}\right) \quad --- (5)$$

[*Speed at any height above the earths surface*]

Now the equation of gravity acceleration for the projectile on the earth's surface is given from modifying the equation of motion (*), such that;

$$-mg = -\frac{GMm}{r_e{}^2}$$

$$\therefore g = \frac{GM}{r_e{}^2} \quad --- (6)$$

From eq.(6) one can say: $GM = gr_e{}^2$, then substituting this value in eq.(5), we get;

$$\vartheta^2 = \vartheta_o{}^2 + 2gr_e{}^2\left(\frac{1}{r_e + x} - \frac{1}{r_e}\right)$$

$$\vartheta^2 = \vartheta_o{}^2 + 2gr_e{}^2\left(\frac{1}{r_e + x} - \frac{1}{r_e}\right)$$

$$\vartheta^2 = \vartheta_o{}^2 + 2gr_e{}^2\left(\frac{r_e - (r_e + x)}{r_e(r_e + x)}\right)$$

$$= \vartheta_o{}^2 + 2gr_e{}^2\left(\frac{r_e - r_e - x}{r_e(r_e + x)}\right)$$

$$= \vartheta_o{}^2 - 2gx\left(\frac{r_e{}^2}{r_e(r_e + x)}\right)$$

$$= \vartheta_o{}^2 - 2gx\left(\frac{1}{\frac{r_e}{r_e} + \frac{x}{r_e}}\right)$$

$$= \vartheta_o^{\ 2} - 2gx\left(\frac{1}{1+\frac{x}{r_e}}\right)$$

$$\therefore \vartheta^2 = \vartheta_o^{\ 2} - 2gx\left(1+\frac{x}{r_e}\right)^{-1} \quad ---(8)$$

$$[\textit{Speed of the projectile with variant gravity acceleration}]$$

In case of ($x \ll r_e \rightarrow \frac{x}{r_e} \cong 0$), then the last equation reduces to the form:

$$\vartheta^2 = \vartheta_o^{\ 2} - 2gx \quad ---(9)$$

$$[\textit{Speed of the projectile with uniform gravitational field}]$$

The maximum height attained by the projectile, is found by setting ($\vartheta = 0$), and solving for x

$$0 = \vartheta_o^{\ 2} - 2gx_{max}\left(1+\frac{x_{max}}{r_e}\right)^{-1}$$

$$2gx_{max}\left(1+\frac{x_{max}}{r_e}\right)^{-1} = \vartheta_o^{\ 2}$$

$$x_{max} = \frac{\vartheta_o^{\ 2}}{2g}\left(1+\frac{x_{max}}{r_e}\right)$$

$$x_{max} = \frac{\vartheta_o^{\ 2}}{2g} + \frac{\vartheta_o^{\ 2}}{2g}\frac{x_{max}}{r_e}$$

$$x_{max}\left(1-\frac{\vartheta_o^{\ 2}}{2gr_e}\right) = \frac{\vartheta_o^{\ 2}}{2g}$$

$$x_{max} = h = \frac{\vartheta_o^{\ 2}}{2g}\left(1-\frac{\vartheta_o^{\ 2}}{2gr_e}\right)^{-1} \quad ---(10)$$

$$[\textit{Maximum height of the projectile}]$$

Again, if $\left({\vartheta_o}^2 \ll 2gr_e \rightarrow \frac{{\vartheta_o}^2}{2gr_e} \cong 0\right)$, then;

$$\boxed{x_{max} = h = \frac{{\vartheta_o}^2}{2g} \quad ---(11)}$$

[$Maximum\ height\ of\ the\ projectile\ with\ low\ initial\ speed$]

To find the value of ϑ_o that make the projectile escape from the earth's gravity, which is called the escape speed, we need to expand the series in parentheses in eq.(10), by using the binomial, as the following;

$$\left(1 - \frac{{\vartheta_o}^2}{2gr_e}\right)^{-1} = 1 - \frac{{\vartheta_o}^2}{2gr_e} + \left(\frac{{\vartheta_o}^2}{2gr_e}\right)^2 - \cdots$$

Now, substituting the result in eq.(10),

$$h = \frac{{\vartheta_o}^2}{2g}\left(1 - \frac{{\vartheta_o}^2}{2gr_e} + \left(\frac{{\vartheta_o}^2}{2gr_e}\right)^2 - \cdots\right)$$

$$h = \frac{{\vartheta_o}^2}{2g} - \left(\frac{{\vartheta_o}^2}{2g}\right)^2 \frac{1}{r_e} + \left(\frac{{\vartheta_o}^2}{2g}\right)^3 \frac{1}{r_e} - \cdots$$

neglecting all terms, except the 1st one, we get;

$$h = \frac{{\vartheta_e}^2}{2g}$$

$$\boxed{\therefore \vartheta_e = \sqrt{2gh}}$$

[$Escape\ velocity$]

Let $h = r_e = 6.4 \times 10^6 m.\ and\ g = 9.8\ m/S^2$

$$\therefore \vartheta_e = \sqrt{2 \times 6.4 \times 10^6 \times 9.8} \cong 11 Km/S$$

In order to find the escape velocity of projectile (36)
we have to expand the binomial in eq (9):

$$X_{max.} = \frac{v_0^2}{2g}\left[1 - \frac{v_0^2}{2g r_e} + \left(\frac{v_0^2}{2g r_e}\right)^2 - \cdots\right]$$

$$= \frac{v_0^2}{2g} - \left(\frac{v_0^2}{2g}\right)^2 \frac{1}{r_e} + \left(\frac{v_0^2}{2g}\right)^3 \frac{1}{r_e^2} - \cdots$$

Neglecting the heigher orders of this series, we get

$$X_{max.} = \frac{v_0^2}{2g}$$

$$v_0^2 = 2g\, X_{max.}$$

If $X_{max} = r_e = 6.4 \times 10^6 m$, $g = 9.8 m/s^2$ the escape velocity (v_e) at Earth's surface will be:

$$v_e = 2(9.8)(6.4 \times 10^6)$$

$$\simeq 11 \text{ km/S}$$

وهو مقدار كبير جدا خصوصاً اذا اخذنا بنظر الاعتبار الزيادة التي تحصل على هذا الرقم بزيادة (h).

* كمثال على سرعة الافلات فان جزيئات O_2 و N_2 في طبقة الاتموسفير تمتلك سرعة افلات تبلغ 0.5 Km/Sec. وهي اقل من سرعة الافلات لذلك يحافظ الغلاف المحيط بالارض بالاوكسجين والنيتروجين على خلاف القمر الذي يمتاز بصغر ضغطه مقارنة بالارض مما لا يسمح للجزيئات اعلاه بالثبات ~~حوله~~ في غلافه.

2-7 The Force as a Function of Velocity only

غالبا ما يحدث ان تكون القوة المؤثرة على الجسم المتحرك هي دالة لسرعة ذلك الجسم . وهذا صحيح في حالة قوة اللزوجة مثلا والتي هي قوة مقاومة من قبل المائع على الجسم المتحرك . ففي حالة مقاومة الموائع فقد وجد بأنه في السرع الواطئة فان المقاومة هي تتناسب (تقريبا) طرديا مع السرعة ، بينما في السرع العالية فان المقاومة تتناسب مع مربع السرعة .

If there are no other forces acting, the diff. equation of motion can be expressed as:

$$F(v) = m\frac{dv}{dt} \quad \text{---} \quad (1)$$

A single Integration yields (t) as a function of v:

$$\int F(v)\,dt = \int m\,dv \quad \rightarrow \text{time as a function of } (v)$$

$$\Rightarrow \boxed{t = \int \frac{m\,dv}{F(v)}} \quad \Rightarrow t \rightarrow t(v)$$

Assuming that we can solve the above equation for (v),

$$\boxed{v = v(t)} \quad \text{velocity as a function of } (t)$$

Second Integration

$$\boxed{\int v(t)\,dt = x(t)} \quad \text{position as a function of } (t)$$

For eq(1):

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

sub. in eq(1)

$$F(v) = m v \frac{dv}{dx}$$

Integrate to obtain (x) as a function of (v):

$$\int dx = \int \frac{m v}{F(v)}\,dv$$

$$\boxed{x = \int \frac{m v}{F(v)}} = x(v) \quad \text{position as a function of } (v)$$

Ex. P.51/2nd book (Horizontal Motion with linear Resist.)

suppose a block is projected with initial velocity ($v_\circ$) on a smoth horizontal plane, but that there is air resistance proporitional to v, that is $F(v) = -cv$, where c is a constant of proporitionality. (The x axis is along the direction of motion). Find 1) the velocity as a function of distance, 2) the equation of time as a function of (v).

sol.

Newoton's 2nd law (as a function of v):

$$F(v) = m\frac{dv}{dt} \quad --- ①$$

the air resistance force is:

$$F(v) = -cv \quad --- ②$$

Then equation of motion is:

$$-cv = m\frac{dv}{dt}$$

$$\int dt = \int \frac{-m}{c}\frac{dv}{v} \longrightarrow t = \frac{-m}{c}\ln v\Big|_{v_\circ}^{v}$$

$$t = \frac{-m}{c}\ln\frac{v}{v_\circ} \qquad \text{time as a function of } v.$$

$$\frac{-tc}{m} = \ln\frac{v}{v_\circ} \longrightarrow e^{-ct/m} = \frac{v}{v_\circ}$$

$$\therefore v = v_\circ e^{-ct/m}$$

2-8 The Force as a Function of Time only

Newton's 2nd law as a function of (t) given as:

$$F(t) = m\frac{dv}{dt}$$

$$dv = \frac{F(t)}{m}dt$$

$$\frac{dx}{dt} = \frac{F(t)}{m}dt = v(t)$$

2nd Integration gives x as a function of t:

$$x = \int v(t)\,dt \qquad \text{where: } \frac{dx}{dt} = v(t)$$

$$\therefore \boxed{x = \int\left[\frac{F(t)}{m}dt\right]dt}$$

Equation of position as a function of (t), For forces being given as a function of (t).

EX. P.53, 2nd Ed. book

A block is initially at rest on a smooth horizontal surface. At time ($t=0$) a constantly increasing horizontal force is applied: $F = ct$. Find the velocity and the displacement as functions of time.

Sol. Newtons 2nd law is given as a function (t):

$$F(v) = m\frac{dv}{dt} \quad --- ①$$

the other increasing applied force is: $F = ct$ --②

$\therefore$ The diff. equation of motion is: $ct = m\dfrac{dv}{dt}$

$$ct\,dt = m\,dv \qquad \div m$$

$$dv = \frac{ct}{m}dt \longrightarrow \int_0^v dv = \frac{c}{m_0}\int^t t\,dt$$

$$v = \frac{c}{m}\,\frac{t^2}{2} \longrightarrow v(t) \text{ equation.}$$

$$\frac{dx}{dt} = \frac{ct^2}{2m} \longrightarrow dx = \frac{ct^2}{2m}\,dt$$

$$\therefore\ x = \frac{ct^3}{6m} \longrightarrow x(t) \text{ equation.}$$

~~2-9 Horizontal Motion with linear Resistance:~~

~~Suppose~~ 2-9 Vertical motion in a Resisting Medium.
Terminal Velocity (P.53, 2nd book)

An object falling vertically through the air or through any fluid is subject to viscous resistance. If the resistance is proporitional to the first power of (v) (linear case), we can express this force as ($-cv$) regardless of the sign of (v), because the resistance is always opposite to the direction of motion. c- is a const. depends on objects shape and size and the viscosity of the fluid.

Let us take x-axis to be positive upward, the diff. eq of motion is then:

$$-mg - cv = m\,\frac{dv}{dt}$$

$$\int_0^t \frac{dt}{m} = \int_{v_0}^{v} \frac{dv}{-(mg+cv)}\,\frac{c}{c} \longrightarrow \frac{t}{m} = -\frac{1}{c}\ln(mg+cv)\Big|_{v_0}^{v}$$

$$\Rightarrow t = \frac{-m}{c}\left[\ln(mg+cv) - (mg+cv_0)\right]$$

$$\therefore\ t = \frac{-m}{c}\ln\left(\frac{mg+cv}{mg+cv_0}\right)$$

نحتاج الى تحويل الداله من معادله زمن (t) بدلاله السرعه (v) الى معادله سرعه (v) بدلاله الزمن (t) :

$$\frac{-tc}{m} = \ln\left(\frac{mg + cv}{mg + cv_0}\right) \qquad \text{by exp.}$$

$$e^{-\frac{tc}{m}} = \frac{mg + cv}{mg + cv_0}$$

$$mg + cv = (mg + cv_0)\, e^{-tc/m}$$

$$\cancel{cv = -mg + mg e^{-tc/m} + cv_0 e^{-tc/m}}$$

$$\boxed{v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{-tc/m}} \quad \text{--- (1)}$$

Velocity as a function of time

if $t \gg m/c$, and velocity approching the limiting value $(-mg/c)$. The limiting velocity of a falling body is called the terminal velocity

سرعه المنتهى هنا هي السرعه التي عندها تكون قوة المقاومه فيها مساويه في المقدار ومعاكسه لقوة وزن الجسم. حيث ان محصله القوة المؤثرة على الجسم صفر - حينها هذه السرعه تدعى terminal speed

$$x - x_0 = \int_0^t v(t)\,dt$$

$$= -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{-tc/m} \quad \text{--- (II)}$$

$\frac{mg}{c} = v_t$, $\frac{m}{c} = \tau$, then: eq (I) becomes:

$$\boxed{v = -v_t + (v_t + v_0)\, e^{-t/\tau}} \quad \text{--- (*)}$$

~~eq (I) becoms (with drivitive)~~ * eq (I) becoms:

$$\frac{dx}{dt} = \frac{-mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{-tc/m}$$

$$\int_{x_0}^{x} dx = -\frac{mg}{c}\int_0^t dt + \frac{mg}{c}\int_0^t e^{-tc/m} dt + v_0 \int_0^t e^{-ct/m} dt$$

A الاول

$$x - x_0 = \frac{-mg}{c} t + \frac{mg}{c}\frac{-c/m}{-c/m}\int e^{-ct/m} dt + v_0\left(\frac{-c/m}{-c/m}\right)\int e^{-ct/m} dt$$

$$= \frac{-mg}{c} t - \frac{m^2 g}{c^2} e^{-ct/m}\Big|_0^t + \left(-v_0 \frac{m}{c}\right) e^{-ct/m}\Big|_0^t$$

$$= \frac{-mg}{c} t + \frac{m^2 g}{c^2}\left(1 - e^{-ct/m}\right) + \frac{v_0 m}{c}\left(1 + e^{-ct/m}\right)$$

~~v_t [illegible]~~

$$x - x_0 = \frac{-mg}{c} t + \left(\frac{m^2 g}{c^2} + \frac{m v_0}{c}\right)\left(1 - e^{-ct/m}\right)$$

$$x = x_0 - v_t t + x_1\left(1 - e^{-ct/m}\right)$$

where; $x_1 = \frac{m^2 g}{c^2} + \frac{m v_0}{c} = g\tau^2 + v_0\tau$

For eq (*), if ($v_0 = 0$) $\therefore$ $v = -v_t + v_t e^{-t/\tau}$

$$v = \left(1 - e^{-t/\tau}\right) v_t$$

if $t = 2\tau$

$$\therefore \boxed{v = \left(1 - e^{-2}\right) v_t}$$

after an interval of ($t = 10\tau$), the speed is equal to (v_t)

Problems

2.1 Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at $x = 0$ and $t = 0$, subject to the following force functions:
(a) $F_x = F_0 + ct$
(b) $F_x = F_0 \sin ct$
(c) $F_x = F_0 e^{ct}$
where F_0 and c are positive constants.

2.2 Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the following force functions:
(a) $F_x = F_0 + cx$
(b) $F_x = F_0 e^{-cx}$
(c) $F_x = F_0 \cos cx$
where F_0 and c are positive constants.

2.3 Find the potential energy function $V(x)$ for each of the forces in Problem 2.2.

2.4 A particle of mass m is constrained to lie along a frictionless, horizontal plane subject to a force given by the expression $F(x) = -kx$. It is projected from $x = 0$ to the right along the positive x direction with initial kinetic energy $T_0 = 1/2\, kA^2$. k and A are positive constants. Find **(a)** the potential energy function $V(x)$ for this force; **(b)** the kinetic energy, and **(c)** the total energy of the particle as a function of its position. **(d)** Find the turning points of the motion. **(e)** Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.5 As in the problem above, the particle is projected to the right with initial kinetic energy T_0 but subject to a force $F(x) = -kx + kx^3/A^2$, where k and A are positive constants. Find **(a)** the potential energy function $V(x)$ for this force; **(b)** the kinetic energy, and **(c)** the total energy of the particle as a function of its position. **(d)** Find the turning points of the motion and the condition the total energy of the particle must satisfy if its motion is to exhibit turning points. **(e)** Sketch the potential, kinetic, and total energy functions. (*Optional:* Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.6 A particle of mass m moves along a frictionless, horizontal plane with a speed given by $v(x) = \alpha/x$, where x is its distance from the origin and α is a positive constant. Find the force $F(x)$ to which the particle is subject.

2.7 A block of mass M has a string of mass m attached to it. A force $\mathbf{F}$ is applied to the string, and it pulls the block up a frictionless plane that is inclined at an angle θ to the horizontal. Find the force that the string exerts on the block.

2.8 Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation

$$\dot{x} = bx^{-3}$$

where b is a positive constant, find the force acting on the particle as a function of x. (*Hint:* $F = m\ddot{x} = m\dot{x}\, d\dot{x}/dx$.)

2.9 A baseball (radius = .0366 m, mass = .145 kg) is dropped from rest at the top of the Empire State Building (height = 1250 ft). Calculate **(a)** the initial potential energy of the baseball, **(b)** its final kinetic energy, and **(c)** the total energy dissipated by the falling baseball by computing the line integral of the force of air resistance along the baseball's total distance of fall. Compare this last result to the difference between the baseball's initial potential energy and its final kinetic energy. *(Hint: In part (c) make approximations when evaluating the hyperbolic functions obtained in carrying out the line integral.)*

2.10 A block of wood is projected up an inclined plane with initial speed v_0. If the inclination of the plane is 30° and the coefficient of sliding friction $\mu_K = 0.1$, find the total time for the block to return to the point of projection.

2.11 A metal block of mass m slides on a horizontal surface that has been lubricated with a heavy oil so that the block suffers a viscous resistance that varies as the $\frac{3}{2}$ power of the speed:

$$F(v) = -cv^{3/2}$$

If the initial speed of the block is v_0 at $x = 0$, show that the block cannot travel farther than $2mv_0^{1/2}/c$.

2.12 A gun is fired straight up. Assuming that the air drag on the bullet varies quadratically with speed, show that the speed varies with height according to the equations

$$v^2 = Ae^{-2kx} - \frac{g}{k} \quad \textit{(upward motion)}$$

$$v^2 = \frac{g}{k} - Be^{2kx} \quad \textit{(downward motion)}$$

Q.2/b

$$F_x = F_0 e^{-cx}$$

$$F_x = m a_x = m \ddot{x}$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \frac{dx}{dt} = \dot{x} \frac{d\dot{x}}{d\cancel{x}}$$

$$m \dot{x} \frac{d\dot{x}}{dx} = F_0 e^{-cx}$$

$$\int \dot{x} \, d\dot{x} = \frac{F_0}{m} \int dx \, e^{-cx}$$

$$\frac{1}{2} \dot{x}^2 = \frac{F_0}{m} \cdot \frac{1}{-c} e^{-cx}$$

$$\boxed{\dot{x} = -\sqrt{\frac{2F_0}{cm} e^{-cx}}}$$