« Countable Sets »

Definition: A function $f: A \longrightarrow B$ is said to be one-to-one (1-1) if $\forall\ a, b \in A$ then $f(a) = f(b)$ iff $a = b$.

Definition: A function $f: A \longrightarrow B$ is said to be onto if $f(A) = B$.

Remark: If f is 1-1 and onto then f is corresspondence and written $(A \sim B)$.

Remark:

$J_n = \{1, 2, \cdots, n\}$ finite

$J = \{1, 2, \cdots\}$ infinite

Definition: A set X is said to be finite if its empty or is equivalent the set J_n for some positive integer. and A set which is not finite is called infinite.

Example: Let $A = \{4, 10, 15, 20\}$

Let $J_4 = \{1, 2, 3, 4\}$

s.t $f(4) = 1$, $f(10) = 2$, $f(15) = 3$, $f(20) = 4$

$\therefore f$ is 1-1 and onto (corresspondence)

then $A \sim J_4$

$\therefore A$ is finite

Definition: The set A is said to be countable if there exist 1-1 and onto function f from A onto J. (i.e $A \sim J$)

Remark: Every finite set is countable.

Example: The set of all integers is countable.

proof: let $f: J \longrightarrow \mathbb{Z}$ be a function defined by $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{-x+1}{2} & \text{if } x \text{ is odd} \end{cases}$

1- To show that f is 1-1

let $a, b \in J$ s.t $a \neq b$

ⓐ if a, b are even

$f(a) = \frac{a}{2}$, $f(b) = \frac{b}{2}$

$\Rightarrow$ since $a \neq b$ $\Rightarrow$ $\frac{a}{2} \neq \frac{b}{2}$

$\therefore f(a) \neq f(b)$

ⓑ if a, b are odd

$f(a) = \frac{-a+1}{2}$, $f(b) = \frac{-b+1}{2}$

since $a \neq b$ $\Rightarrow$ $\frac{-a+1}{2} \neq \frac{-b+1}{2}$

$\therefore f(a) \neq f(b)$

Then f is 1-1

2:- To show that f is onto

$\forall\ b \in \mathbb{Z}$ T.P $\exists\ a \in J$ s.t onto

(a) If a is even then $f(a) = \frac{a}{2}$

$$\Rightarrow \frac{a}{2} = b \Rightarrow a = 2b$$

$\therefore\ f(a) = \frac{a}{2} = \frac{2b}{2} = b$

$\therefore\ f(a) = b$

(b) If a is odd then $f(a) = \frac{-a+1}{2}$

$$\Rightarrow \frac{-a+1}{2} = b \Rightarrow a = -2b+1$$

$\therefore\ f(a) = \frac{-a+1}{2} = \frac{-(-2b+1)}{2} = \frac{2b-1+1}{2} = b$

$\therefore\ f(a) = b$

$\therefore\ f$ is onto

$\therefore\ \mathbb{Z} \sim J$

$\therefore\ \mathbb{Z}$ is countable

Definition : A set A which is not countable and not finite is called uncountable.

Example: The set $\mathbb{R}$ of all real numbers is uncountable

proof: Let $S \subseteq \mathbb{R}$ and S is countable

T.P $S \neq \mathbb{R}$

let $S = \{a_1, a_2, \dots, a_n, \dots\}$

let I_1 be interval s.t $|I_1| < 1$ and $a_1 \notin I_1$

let I_2 be interval s.t $|I_2| < \frac{1}{2}$ and $a_2 \notin I_2$, $I_1 \supseteq I_2$

$\vdots$

let I_3 be interval s.t $|I_n| < \frac{1}{n}$ and $a_n \notin I_n$, $I_{n-1} \supseteq I_n$

we have $\bigcap_{n=1}^{\infty} I_n = \{x\}$

$\therefore x \in I_n, \forall n$

since $a_n \notin I_n \Rightarrow a_n \neq x$

$\therefore x \notin S \Rightarrow \mathbb{R} \neq S$

$\therefore \mathbb{R}$ is not countable.

Definition: Let X be a set. A function $f: J \to X$ is called sequence in X.

Remarks: 1- for each $n \in J$ we denoted by the value $f(n)$ by $\{a_n\}$

2- If $X = \mathbb{R}$ then the sequence is called sequnce of real numbers.

3- If X is countable set then X is range of sequnce.

Theorem: Every infinite subset of countable set is countable.

proof: Let X be a countable set and A be an infinite subset of X.

$\therefore X \sim J$ (by def. of countable) Thus $J \sim X$

$\therefore X$ is arange of sequence

elements of X can which as written as

$x_1, x_2, \ldots$ which are distinct element.

let n_1 be the smallest positive integer $\ni X_{n_1} \in A$

let n_2 " " " " " $\ni n_2 > n_1$ and $X_{n_2} \in A$

let n_3 " " " " " $\ni n_3 > n_2$ and $X_{n_3} \in A$

$\vdots$

let n_k " " " " " $\ni n_k > n_{k-1}$ and $X_{n_k} \in A$

let $f : J \longrightarrow A$ defined by $f(k) = X_{n_k}$

$\therefore f$ is 1-1 and onto

$\therefore J \sim A \Rightarrow A \sim J$

$\therefore A$ is countable.

Theorem: Let $\{E_n\}$ be a sequence of countable sets and let $S = \bigcup_{n=1}^{\infty} E_n$, Then S is countable.

proof: since E_n is countable for each n

$\therefore J \sim E_n$ for each n

elements of E_n can be arranged is a sequence of distinct elements of

$E_1 = \{ x_{11}, x_{12}, x_{13}, \cdots \}$

$E_2 = \{ x_{21}, x_{22}, x_{23}, \cdots \}$

$E_3 = \{ x_{31}, x_{32}, x_{33}, \cdots \}$

$S = E_1 \cup E_2 \cup E_3 \cup \cdots$

Arrange elements of S in a sequence as follow $S = \{ x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, \cdots \}$

if same element apear more than one we choose one of them only

since E_1 is infinite and $E_1 \subseteq S$

$\therefore S$ is infinite

Define a function $f : J \longrightarrow S$ as follow

$f(1) = x_{11}$, $f(2) = x_{21}$, $f(3) = x_{12}$, $\cdots$

Then f is 1-1 and onto

$\therefore J \sim S$

Thus S is countable.

<u>Corollary</u>: The set of all rational number is countable.

<u>proof</u>: Let $Q = \bigcup_{n=1}^{\infty} E_n$ where $E_n = \{ \frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n}, \cdots \}$

let $f : \mathbb{Z} \longrightarrow E_n$ s.t $f(m) = \frac{m}{n}$

$\therefore$ f is 1-1 and onto

$\therefore$ $\mathbb{Z} \sim E_n$

since $\mathbb{Z}$ is countable then E_n is countable

By theorem above $\Rightarrow \bigcup_{n=1}^{\infty} E_n$ is countable

$\therefore$ Q is countable.

Corollary : The set of all irrational number is uncountable.

proof: Suppose Q^c is countable

$\therefore$ $Q \cup Q^c$ is countable

But $Q \cup Q^c = \mathbb{R}$

and $\mathbb{R}$ is is uncountable C!

$\therefore$ Q^c is uncountable.