Chapter 2 : Analytic Functions

Def. Let be S be a set of complex numbers . A function f defined on S a rule which assigns to each z a complex number w . The number w is called a value of f at z and denoted by $f(z)$, that is $w = f(z)$. The set S is called the domain of definition of f (D_f) .

Ex. $w = z^2 + 2z - 7$. the domain of def. is $\mathbb{C}$.
Ex. $w = \frac{1}{z^2+4}$. $\quad D_f = \{\mathbb{C}\backslash \pm 2i\}$.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$,
$\therefore u + iv = f(x + iy)$, each of the real number u & v depends on the real variables x & y ,
$\therefore w = f(z) = u(x, y) + iv(x, y)$.

Ex. $w = f(z) = z^2 \rightarrow w = x^2 - y^2 + 2ixy \rightarrow u + iv = (x^2 - y^2) + i(2xy)$
$u = x^2 - y^2$, $v = 2xy$.

Limits

Def. $\lim_{z\rightarrow z_0} f(z) = w_0$ mean $\forall \epsilon > 0$, $\exists\, \delta > 0$ *such that*
$|f(z) - w_0| < \epsilon$ *wherever* $0 < |z - z_0| < \delta$. draw
Ex. Prove that $\lim_{z\rightarrow 2i-1}(2z + 3) = 4i + 1$?
Sol. $\forall \epsilon > 0$ *we must find* $\delta > 0$ *such that*
$0 < |z - (2i - 1)| < \delta \rightarrow |2z + 3 - (4i + 1)| < \epsilon$.
By the def. $|f(z) - w_0| < \epsilon$

$$|2z + 3 - (4i + 1)| < \epsilon \rightarrow |2z - 4i + 2| < \epsilon$$

$$\rightarrow \left|2\big(z - (2i - 1)\big)\right| < \epsilon \rightarrow |z - (2i - 1)| < \frac{\epsilon}{2} = \delta \,.$$

Theorems of Limits

1- When a limit of a function f exist at apoint z_0 , it is unique .
2- Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ & $w_0 = u_0 + iv_0$, then $\lim_{z\rightarrow z_0} f(z) = w_0 \Leftrightarrow \lim_{(x,y)\rightarrow(x_0,y_0)} u(x, y) = u_0$,
$\Leftrightarrow \lim_{(x,y)\rightarrow(x_0,y_0)} v(x, y) = v_0$.

Suppose $\lim_{z\rightarrow z_0} f(z) = w_0$ & $\lim_{z\rightarrow z_0} g(z) = w_1$, then :
3- $\lim_{z\rightarrow z_0}[f(z) \pm g(z)] = w_0 \pm w_1$.

4- $\lim_{z\to z_0}[f(z).g(z)] = w_0 . w_1$.

5- $\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{w_0}{w_1}$, $w_1 \neq 0$.

6- If $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \dots \dots . + a_n z^n$,

when $a_0, a_1 a_2, \dots .., a_n$ are complex constants, then $\lim_{z\to z_0} P(z) = P(z_0)$.

7- If $\lim_{z\to z_0} f(z) = w_0$, *then* $\lim_{z\to z_0} |f(z)| = |w_0|$.

Ex. Find the $\lim_{z\to 1+i}\frac{5z+1}{5z-1} = \frac{\lim_{z\to 1+i} 5z + \lim_{z\to 1+i} 1}{\lim_{z\to 1+i} 5z - \lim_{z\to 1+i} 1} = \frac{5(1+i)+1}{5(1+i)-1} = \frac{6+5i}{4+5i} = \frac{50}{41} + \frac{i}{41}$.

Exercises :

I. Prove that

1- $\lim_{z\to 0}\frac{(\bar{z})^2}{z} = 0$.

2- $\lim_{z\to 1-i}[x + i(2x + y) = 1 + i$.

II. Prove that $\lim_{z\to 0}\frac{\bar{z}}{z}$ is not exist.

III. Find the following limit $\lim\limits_{z\to 2e^{i\frac{\pi}{3}}} \frac{z^3+8}{z^4+4z^2+16}$?

Continuity

Def. A function f is <u>continuous</u> at point z_0 iff :

$1 -$ $\lim_{z\to z_0} f(z)$ exist ;

$2 -$ $f(z_0)$ exist ;

$3 -$ $\lim_{z\to z_0} f(z) = f(z_0)$.

Def. A function f is said to be <u>continuous in D</u>, means its continuous at each point of D.

Ex. $1 - f(z) = z$ *is continuos* , $2 - f(z) = \frac{1}{z}$ *is discontinuos* ,

$3 - f(z) = xy^2 - i(2x - y)$ *is continuos* .

Theorems

1. *Let* $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$. Then $f(z)$ *is continuos iff* $u(x,y)$ *and* $v(x,y)$ *are continuos at* $z = z_0$.

2. $Let\ f\ and\ g\ are\ continuos\ fun\ ctions\ , then$

   $i. f \pm g\ \ cont.$

$ii. f . g\ \ cont.$

$iii. \frac{f}{g}\ \ cont.\ ,\ g \neq 0$

$iv.\ fog\ \ cont.\ .$

Q. Does the function $f(z) = \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$ continuous at $z = i$?

Derivatives

Def. Let f a function whose domain of definition contains a neighborhood of a point z_0.

We define the derivative of f at z_0 ,written by $\acute{f}(z)$, by the equation :

$$\acute{f}(z) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \ldots\ldots\ldots..(1)$$

provided the limit here exists.

The function f is said to be differentiable at z_0, when its derivative at z_0 exist.

Expressing z in equation (1) in terms of the new complex variable $\Delta z = z - z_0$,

We can write the equation (1) as

$$\acute{\boldsymbol{f}}(\boldsymbol{z}) = \mathbf{lim}_{\Delta \boldsymbol{z} \to \mathbf{0}} \frac{\boldsymbol{f}(\boldsymbol{z}+\Delta \boldsymbol{z})-\boldsymbol{f}(\boldsymbol{z})}{\Delta \boldsymbol{z}} \ldots\ldots\ldots..(2)$$

Or $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ ....(3) , where $\Delta w = f(z + \Delta z) - f(z)$.

Ex. By the def. , find the $\acute{f}(z)$ for $f(z) = z^2$?

Sol. $\acute{f}(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2+2z\Delta z+(\Delta z)^2-z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$

Q1. Does $\acute{f}(z)$ exist at $z = 0$ for $f(z) = |z|^2$?

Q2. Prove that $\acute{f}(z)$ is not exist for $f(z) = \overline{z\ , (Im \neq 0)}$?

Theorem : If f has a derivative at z_0 , then it is continuous at z_0 .

Theorem : If a function f is differentiable at z_0, then it is continuous at z_0 .

Proof:

$\because f\ diff.\ at\ z_0 \to \acute{f}(z_0)$ exist.

$$\therefore \lim_{\Delta z \to 0} [f(z_0 + \Delta z) - f(z_0) = \lim_{\Delta z \to 0} \frac{[f(z_0+\Delta z)-f(z_0)}{\Delta z} . \Delta z = \lim_{\Delta z \to 0} \frac{[f(z_0+\Delta z)-f(z_0)}{\Delta z} . \lim_{\Delta z \to 0} \Delta z$$

$$= \acute{f}(z_0) . 0 = 0\ .$$

$\therefore \lim_{\Delta z \to 0} [f(z_0 + \Delta z) = f(z_0)$

$\therefore \lim_{z \to z_0} f(z) = f(z_0) \to \therefore f(z)\ is$ continuous at z_0 ■

Note *: The converse of this theorem is not always corect.*

Ex. $f(z) = \bar{z}$, $Im(z) \neq 0 \to \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f\overline{(z+\Delta z)}-\bar{z}}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$, $\Delta z = \Delta x + i\Delta y$ *and* $\overline{\Delta z} = \Delta x - i\Delta y$

if Δz *is Re* $\to \Delta y = 0$, $\Delta z = \Delta x$ *and* $\overline{\Delta z} = \Delta x$

$\therefore \lim_{\Delta z = \Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$,

if Δz *is Im* $\to \Delta x = 0$, $\Delta z = i\Delta y$ *and* $\overline{\Delta z} = -i\Delta y$

$\therefore \lim_{\Delta z = \Delta x \to 0} \frac{-i\Delta y}{i\Delta y} = -1$

$\therefore \acute{f}(z)$ *is not exist even* $f(z)$ *is continuous* .

Differentiation Formulas

1- $\frac{d}{dz}(c) = 0$, c *constant*.

2- $\frac{d}{dz}(z) = 1$

3- $\frac{d}{dz}[c.f(z)] = c.\acute{f}(z)$

4- $\frac{d}{dz}[f(z) \pm g(z)] = \acute{f}(z) \pm \acute{g}(z)$

5- $\frac{d}{dz}[f(z).g(z)] = f(z).\acute{g}(z) + g(z)\acute{f}(z)$

6- $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z).\acute{f}(z)-f(z).\acute{g}(z)}{[g(z)]^2}$, $g(z) \neq 0$

7- $\frac{d}{dz} z^n = nz^{n-1}$

8- *Let* $f(z) = g\big(h(z)\big) \to \acute{f}(z) = \acute{g}\big(h(z)\big).\acute{h}(z)$.

Ex. Does $\acute{f}(z)$ exist at $z_0 = 0$ for $f(z) = |z|^2$?

Solution: $\acute{f}(z) = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z})-|z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z\bar{z}+z\overline{\Delta z}+\bar{z}\Delta z+\Delta z\overline{\Delta z}-z\bar{z}}{\Delta z}$

$= \lim_{\Delta z \to 0} z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$

$\therefore \acute{f}(z)$ is exist at $z_0 = 0$ and equal to zero.

Q. Does $\acute{f}(z)$ exist at any point ? No, check ???

The Cauchy-Riemann Equations $(\boldsymbol{C-R\ \ Eq.})$

Def. Let $f(z) = u(x,y) + iv(x,y),\ where\ \ z = x + iy.$ The equations

$$\boldsymbol{u_x(x,y) = v_y(x,y)}\ \ ------(1)$$

$$\boldsymbol{u_y(x,y) = -v_x(x,y)}\ \ -----(2),$$

are called the Cauchy – Riemann equations or Cauchy – Riemann conditions at $z = x + iy$.

Theorem $(\boldsymbol{Cauchy - Riemann})$

Suppose $f(z) = u(x,y) + iv(x,y),\ where\ \ z_0 = x_0 + iy_0.$ Then $\acute{f}(z_0)$ exists iff $u_x(x,y), u_y(x,y), v_x(x,y)\ and\ \ v_y(x,y)$ are exists and satisfy C-R Eq. 1 and 2, moreover $\acute{f}(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$.

Ex. Let $f(z) = z^2 = x^2 - y^2 + 2ixy \rightarrow\ \ u = x^2 - y^2$, $v = 2xy$

$u_x = 2x$, , $u_y = -2y$, $v_x = 2y$, $v_y = 2x$

$\therefore u_x = v_y\ \ and\ \ v_x = -u_y$,

$\therefore\ \ C - R\ Eq.\ are\ hold.$

$\therefore\ \acute{f}(z) = u_x + iv_x = 2x + 2iy$ or $\acute{f}(z) = v_y - iu_y = 2x + 2iy$.

The Cauchy-Riemann Equations in Polar forms

Let $z = x + iy = re^{i\theta}$, $x = rcos\theta$, $y = sin\theta$, and let $f(z) = u(x,y) + iv(x,y)$,

Using a chain rule and the def. of the derivative:

1- $For\ \ r$, $\acute{f}(z).\frac{\partial z}{\partial r} = u_r + iv_r$

since $z = re^{i\theta}\ \rightarrow \frac{\partial z}{\partial r} = e^{i\theta}$

$\therefore\ \acute{f}(z).e^{i\theta} = u_r + iv_r$ -------------(1)

2- For θ , $\acute{f}(z).\frac{\partial z}{\partial \theta} = u_\theta + iv_\theta$, $\frac{\partial z}{\partial \theta} = rie^{i\theta}$

$\therefore\ \acute{f}(z).rie^{i\theta} = u_\theta + iv_\theta$ ----------(2)

$\acute{f}(z).ire^{i\theta} = iru_r - rv_r$ ------------($\acute{1}$) (x ir)

i.e. from 1´ & 2 $\rightarrow iru_r - rv_r = u_\theta + iv_\theta$

$\boldsymbol{u_\theta = -rv_r}$

$\boldsymbol{v_\theta = ru_r}$ C-R Eq. in polar form.

Now what about the derivative $\acute{f}(z)$ in polar form ?

$\acute{f}(z) = e^{-i\theta}[U_r(r,\theta) + iV_r(r,\theta)]$ ------------ (3)

$= \frac{1}{r}\, e^{-i\theta}[V_\theta(r,\theta) - iU_\theta(r,\theta)]$ -----------(4).

Ex. Consider the function $g(z) = \sqrt{r}\,.e^{i\frac{\theta}{2}}$, $(r > 0\,,\ 0 < \theta \le \pi)$. show that $g(z)$ Has a derivative at each point in its domain of definition and $\acute{g}(z) = \frac{1}{2g(z)}$?

Sol. $g(z) = r^{\frac{1}{2}}\left(cos\frac{\theta}{2} + isin\frac{\theta}{2}\right) = r^{\frac{1}{2}}\,cos\frac{\theta}{2} + ir^{\frac{1}{2}}\,sin\frac{\theta}{2}$.

$\therefore u = r^{\frac{1}{2}}\,cos\frac{\theta}{2}$, $v = r^{\frac{1}{2}}\,sin\frac{\theta}{2}$

$$U_r = \frac{1}{2}r^{-\frac{1}{2}}\,cos\frac{\theta}{2} \quad , \quad U_\theta = -\frac{1}{2}r^{\frac{1}{2}}\,sin\frac{\theta}{2}$$

$$V_r = \frac{1}{2}r^{-\frac{1}{2}}\,sin\frac{\theta}{2} \quad , \quad V_\theta = \frac{1}{2}r^{\frac{1}{2}}\,cos\frac{\theta}{2}$$

$$\therefore\ C - R\ Eq.\ are\ holds\,, \quad \therefore\ \acute{g}(z)\ exist.$$

$$\acute{g}(z) = e^{-i\theta}[U_r + iV_r] = e^{-i\theta}\left[\frac{1}{2}\,r^{-\frac{1}{2}}cos\frac{\theta}{2} + i\frac{1}{2}r^{-\frac{1}{2}}\,sin\frac{\theta}{2}\right]$$

$$= e^{-i\theta}\frac{1}{2\sqrt{r}}\left(cos\frac{\theta}{2} + i\,sin\frac{\theta}{2}\right) = e^{-i\theta}\frac{1}{2\sqrt{r}}\,e^{i\frac{\theta}{2}} = \frac{1}{2\sqrt{r}}\,e^{-i\frac{\theta}{2}} = \frac{1}{2\sqrt{r}e^{-i\frac{\theta}{2}}}$$

$$= \frac{1}{2g(z)}.$$

Analytic Function

Def. A function f of the complex variable z is **analytic** at a point z_0 if its derivative exists not only at z_0 but at each point of a neighborhood of z_0 .

Def. A function f is said to be analytic in the D if it is analytic at each point Of the domain D .

Ex. $f(z) = |z|^2$ is not analytic at any point except at $z = 0$

$f(z) = 3z - \frac{1}{3}z + 1$ is analytic everywhere .

Def. A function f is **entire** if it is analytic at each point of the complex plane.

Ex. $f(z) = e^z$ is entire,

$f(z) = \sum_{k=0}^{n} a_k z^k$ is entire .

Def. If a function f is not analytic at z_0 but it is analytic at each point of neighborhood of z_0 , then z_0 is called a **singular point** or a **singularity of f** .

Ex. $f(z) = \frac{sinz}{z-2}$, the singular point is $z = 2$.

Harmonic Functions

Def. A real – valued function h of two real variables $x \; and \; y$ is said to be **harmonic** in a given domain of the $xy - plane$ if through that domain it has contains first and second partial derivatives and satisfies the partial differential equation :

$$\boldsymbol{h_{xx}(x, y) + h_{yy}(x, y) = 0} ,$$

this equation is called **Laplace's equation** .

Def. If two given functions $U(x, y) \quad and \quad V(x, y)$ are harmonic in a domain D and their first partial derivatives satisfy the $C - R\ Eq$. Through D .
we say that V is the **harmonic conjugate** of U .

Theorem. A function $f(z) = U(x, y) + iV(x, y)$ is analytic in a domain D iff V is the harmonic conjugate of U.

Ex. If $u(x, y) = y^3 - 3x^2y$ is the Re. part of the analytic function $f(z)$, find the Im. Part (harmonic conjugate)?

Sol. $U_x = -6xy = V_y$, $U_y = 3y^2 - 3x^2$,

$V_y == -6xy \rightarrow V = \int -6xydy \rightarrow V = -3xy^2 + F(x)$

$V_x = -3y^2 + \acute{F}(x) = -U_y \quad \rightarrow -3y^2 + \acute{F}(x) = -3y^2 + 3x^2 \rightarrow \acute{F}(x) = 3x^2$

$\therefore F(x) = x^3 + c.$

$\therefore V = -3xy^2 + x^3 + c$.

Ex. Prove that $f(x, y) = e^x\, cosy$ is a harmonic function ?

Sol. $\frac{\partial f}{\partial x} = e^x\, cosy \rightarrow \frac{\partial^2 f}{\partial x^2} = e^x\, cosy$

$\frac{\partial f}{\partial y} = -e^x\, siny \rightarrow \frac{\partial^2 f}{\partial y^2} = -e^x\, cosy$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x\, cosy + (-e^x\, cosy) = 0.$

$\therefore \ f \ \ is \ harmonic$.

Ex. $If \ U(x,y) = e^x \ cosy$, find the harmonic conjugate $V(x,y)$?

Sol. $U_x = e^x \ cosy = V_y \quad , \ U_y = -e^x \ siny$

$V_y = e^x \ cosy \ \rightarrow V = e^x \ siny + F(x)$

$V_x = e^x \ siny + \acute{F}(x) = -U_y \ \rightarrow e^x \ siny + \acute{F}(x) = -(-e^x \ siny)$

$\acute{F}(x) = 0 \ \rightarrow \ F(x) = c. \quad , \quad \therefore V = e^x \ siny + c.$

Laplace's Equation in Polar form

$U_{xx} + U_{yy} = 0 \ \approx \boldsymbol{r^2 U_{rr} + r U_r + U_{\theta\theta} = 0}$ -----------(1)

$V_{xx} + V_{yy} = 0 \ \approx \boldsymbol{r^2 V_{rr} + r V_r + V_{\theta\theta} = 0}$ -----------(2) Laplace's Eq. in polar form.

Ex. Let $g(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \ \rightarrow U = \sqrt{r}\ \cos\frac{\theta}{2} \quad , \ V = \sqrt{r}\ \sin\frac{\theta}{2}$

$U_r = \frac{1}{2}\, r^{-\frac{1}{2}}\ cos\frac{\theta}{2} \ \rightarrow \ U_{rr} = \frac{-1}{4}\, r^{-\frac{3}{2}} cos\frac{\theta}{2}$

$U_\theta = \frac{-1}{2}\, r^{\frac{1}{2}}\ sin\frac{\theta}{2} \ \rightarrow \ U_{\theta\theta} = \frac{-1}{4}\, r^{\frac{1}{2}} cos\frac{\theta}{2}$

$\therefore r^2\left(\frac{-1}{4}\, r^{-\frac{3}{2}} cos\frac{\theta}{2}\right) + r\left(\frac{1}{2}\, r^{-\frac{1}{2}}\ cos\frac{\theta}{2}\right) - \frac{-1}{4}\, r^{\frac{1}{2}} cos\frac{\theta}{2} = 0$

$\therefore laplace's Eq. is \ hold. \quad \therefore \ U \ is \ harmonic$.

Q. Let $g(z) = r^2\, e^{i2\theta}$, prove that V is harmonic ? (H.W).