Chapter 3 : The Elementary Functions

1. The Exponential Function e^z

Def. Let $z = x + iy \rightarrow e^z = e^x e^{iy} = e^x(cosy + isiny)$ ---------(1)

$\therefore \boldsymbol{e^z = e^x cosy + ie^x siny}$ ----------(2)

<u>Properties of e^z</u>

1. $e^z \neq 0 \ , \forall z \in \mathbb{C}$.
2. $e^{z_1}.e^{z_2} = e^{z_1+z_2}$.
3. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$.
4. $(e^z)^n = e^{nz}$.
5. $f(z) = e^z$ *is entire function and* $\acute{f}(z) = e^z = f(z)$.
6. *In polar form* $e^z = \rho(cos\emptyset + isin\emptyset), where \ \rho = |e^x| = e^x, \emptyset = arge^z$.

   Therefore we find from (1) that $w = e^z = \rho(cos\emptyset + isin\emptyset)$ is the Image of $z = log\rho + i\emptyset$

Ex. Find all z from which $e^z = -1$?

Sol. $-1 = \cos(\pi + 2n\pi) + isin(\pi + 2n\pi), \ n = 0, \pm 1, \pm 2, \dots \dots$

$\therefore -1 = e^{i(\pi+2n\pi)} \rightarrow e^z = e^{i(\pi+2n\pi)}$

$\therefore z = i(\pi + 2n\pi) \ , \ n = 0, \pm 1, \pm 2, \dots \dots$.

Ex. Solve the equation $e^{2z-1} = 1$?

Sol. $1 = cos2n\pi + isin2n\pi \ , \quad n = 0, \pm 1, \pm 2, \dots \dots$

$e^{2z-1} = e^{i2n\pi} \rightarrow 2z - 1 = i2n\pi \rightarrow z = \frac{1}{2} + in\pi \quad , \ n = 0, \pm 1, \pm 2, \dots \dots$.

Q. Write $\left|e^{2z+i}\right|$ *and* $\left|e^{iz^2}\right|$ *in terms of* x *and* y, then show that

$\left|e^{2z+i} + e^{iz^2}\right| \leq e^{2x} + e^{-2xy}$.

2- The Trigonometric Functions

Def. $sinz = \frac{e^{iz}-e^{-iz}}{2i}$, $cosz = \frac{e^{iz}+e^{-iz}}{2i}$, $tanz = \frac{sinz}{cosz}$, $secz = \frac{1}{cosz}$

$cscz = \frac{1}{sinz}$, $cotz = \frac{cosz}{sinz}$.

Theorem 1.

$sinz$ and $cosz$ are entire functions, and

$\frac{d}{dz} sinz = cosz$, $\frac{d}{dz} cosz = -sinz$.

Theorem 2. tanz is analytic function in all z except z which make $cosz = 0$.

$$\frac{d}{dz} tanz = sec^2 z \quad , \quad \frac{d}{dz} secz = secz.tanz$$

$$\frac{d}{dz} cscz = -cscz.cotz \quad , \quad \frac{d}{dz} cotz = -csc^2 z.$$

Another way to write $sinz$ and $cosz$:

$$sinz = \frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} = \frac{e^{-y+ix}-e^{y-ix}}{2i}$$

$$= \frac{e^{-y}}{2i}[cosx + isinx] - \frac{e^{y}}{2i}[cosx - isinx]$$

$$= sinx[\frac{e^{y}+e^{-y}}{2}] + icosx[\frac{e^{y}-e^{-y}}{2}]$$

$$\therefore \boldsymbol{sinz = sinx.coshy + icosx.sinhy.}$$

Similarly

$$\boldsymbol{cosz = cosx.coshy - isinx.sinhy.}$$

Properties of Trigonometric Functions:

1- $|sinz|^2 = sin^2 x + sinh^2 y$
2- $|cosz|^2 = cos^2 x + sinh^2 y$
3- $Sin^2 z + cos^2 z = 1$
4- $Sin(z_1 \pm z_2) = sinz_1.cosz_2 \pm cosz_1.sinz_2$
5- $cos(z_1 \pm z_2) = cosz_1.cosz_2 \mp sinz_1.sinz_2$
6- $\sin(-z) = -sinz$, $\cos(-z) = cosz$
7- $sinz = 0$ iff $z = n\pi$, $n = 0, \pm 1, \pm 2, \dots\dots$
8- $cosz = 0$ iff $z = (n + \frac{1}{2})\pi$, $n = 0, \pm 1, \pm 2, \dots\dots$

Proof of 7: $sinz = 0 \rightarrow sinz = sinx.coshy + icosx.sinhy = 0 + i0$

$sinx.coshy = 0$ ----------(1)

$cosx.sinhy = 0$ ----------(2)

From (1) either $sinx = 0 \;\; or \;\; coshy = 0$, $coshy \neq 0 \;\; because \; always$

$$coshy = \frac{e^y + e^{-y}}{2} \geq 1$$

$$\therefore sinx = 0 \rightarrow x = n\pi \;\; , \qquad n = 0, \pm1, \pm2, \dots \dots$$

Substitute x in (2)

$\cos(n\pi)\,.sinhy = 0 \rightarrow (-1)^n.sinhy = 0 \rightarrow sinhy = 0 \rightarrow y = 0$

$$\therefore z = n\pi + i.0 = n\pi \;\; , \;\; n = 0, \pm1, \pm2, \dots \dots$$

Conversely (H.W)

Ex. Find all the roots of the equation $sinz = cosh4$?

Sol. $cosh4 = \frac{e^4+e^{-4}}{2} > 1$

$$sinz = sinx.coshy + icosx.sinhy = cosh4 + i.0$$

$sinx.coshy = cosh4$ ---------(1)

$cosx.sinhy = 0$ --------(2)

From (2) either $cosx = 0 \;\; or \;\; sinhy = 0$

$If \;\; sinhy = 0 \rightarrow y = 0$

Substitute y in (1) , we get

$sinx.cosh0 = cosh4 \rightarrow sinx = cosh4 > 1$ contradiction because $-1 \leq sinx \leq 1$

$\therefore cosx = 0 \rightarrow x = (n + \frac{1}{2})\pi \;\; , n = 0, \pm1, \pm2, \dots \dots$

Substitute x in (1) , we get

$\text{Sin}[(n+\frac{1}{2})\pi].coshy = cosh4 \rightarrow (-1)^n\ .coshy = cosh4 \rightarrow y = \pm 4$

$\therefore z = (n+\frac{1}{2})\pi \pm i4$.

3- Hyperbolic Functions

Def. $sinhz = \frac{e^z - e^{-z}}{2}$, $coshz = \frac{e^z + e^{-z}}{2}$, $tanhz = \frac{sinhz}{coshz}$,

$cschz = \frac{1}{sinhz}$, $sechz = \frac{1}{coshz}$, $cothz = \frac{1}{tanhz}$.

Theorem.1.

$\frac{d}{dz} sinhz = coshz$, $\frac{d}{dz} coshz = sinhz$, $\frac{d}{dz} tanhz = sech^2 z$

$\frac{d}{dz} cothz = -csct^2 z$, $\frac{d}{dz} sechz = -sechz.tanhz$, $\frac{d}{dz} cschz = -cscz.cothz$.

Real and Imaginary Parts

$$\boldsymbol{sinhz = sinhx.cosy + i\ coshx.siny.}$$

$$\boldsymbol{coshz = coshx.cosy + i\ sinhx.siny.}$$

Properties of Hyperbolic Functions:

1- $cosh^2 z - sinh^2 z = 1$
2- $Sinh(z_1 \pm z_2) = sinhz_1.coshz_2 \pm coshz_1.sinhz_2$
3- $cosh(z_1 \pm z_2) = coshz_1.coshz_2 \mp sinhz_1.sinhz_2$
4- $\sinh(-z) = -sinhz$, $\cos h(-z) = coshz$
5- $\sinh(iz) = isinz$, $\cos h(iz) = cosz$
6- $|sinhz|^2 = sinh^2 x + sin^2 y$
7- $|coshz|^2 = sinh^2 x + cos^2 y$
8- $\sinh(z + 2i\pi) = sinhz$, $\cosh(z + 2i\pi) = coshz$, $tanh(z + 2i\pi) = tanhz$
9- $sinhz = 0$ iff $z = ni\pi$, $n = 0, \pm 1, \pm 2, \dots \dots$
10- $coshz = 0$ iff $z = \left(n + \frac{1}{2}\right) i\pi$, $n = 0, \pm 1, \pm 2, \dots \dots$

Ex. Solve the equation $coshz = \frac{1}{2}$?

Sol. $coshz = coshx.cosy + i\, sinhx.siny = \frac{1}{2}$

$coshx.cosy = \frac{1}{2}$ ---------(1)

$sinhx.siny = 0$ ---------(2)

From (2) : 1- $if\ sinhx = 0 \rightarrow x = 0$, Substitute x in (1) , we get

$$\cosh(0).cosy = \frac{1}{2} \rightarrow cosy = \frac{1}{2} \rightarrow y = \frac{\pi}{3} + 2n\pi, \quad n = 0, \pm1, \pm2, \dots\dots$$

$$\therefore z = \left(\frac{1}{3} + 2n\right)\pi i \ , \quad n = 0, \pm1, \pm2, \dots\dots$$

2. If $siny = 0 \rightarrow y = 0 + n\pi$, Substitute x in (1) , we get

$coshx.\cos(n\pi) = \frac{1}{2} \rightarrow (-1)^n.coshx = \frac{1}{2} \rightarrow coshx = \frac{e^z+e^{-z}}{2} = (-1)^n.\frac{1}{2}$

$if\ n\ is\ odd \ - \ impossible$

$$if\ n\ is\ even \rightarrow \frac{e^x + e^{-x}}{2} = \frac{1}{2} \rightarrow [e^x + e^{-x} = 1] * e^x \rightarrow e^{2x} - e^x + 1 = 0$$

$$Let\ \ u = e^x \rightarrow \ u^2 - u + 1 = 0 \rightarrow u = \frac{1 \pm \sqrt{1-4}}{2}$$

$$\rightarrow \frac{1 \pm i\sqrt{3}}{2} \quad - impossible$$

$So\ \ z = z = \left(\frac{1}{3} + 2n\right)\pi i$, $n = 0, \pm1, \pm2, \dots\dots$.

Q. 1- Find all $z\ for\ sinhz = 1$?

2- Solve the eq. $coshz = -2$?

4- The Logarithmic Functions

Def. Let $z = re^{i\theta}$, $define\ logz = lnr + i\theta$ called Logarithmic function, where $r = |z|$, $\theta = \arg z$, if $\emptyset$ is the principle value of arg z , $(-\pi < \emptyset \leq \pi)$, we can write $\theta = \emptyset + 2n\pi$, $n = 0, \pm1, \pm2, \dots\dots$,

$\therefore logz = \ln r + i(\emptyset + 2n\pi)$, $n = 0, \pm1, \pm2, \dots\dots$ the general logarithm

$Logz = \ln r + i\theta$ the principle value of $logz$.

Therefore , we say that : $w = logz \ \ iff \ \ z = e^w$.

Theorem . The function $logz$ is analytic in the domain $[r > 0 , -\pi < \emptyset \leq \pi]$.

Furthermore , if $z = re^{i\theta}$ $, \frac{d}{dz} logz = e^{-i\theta}\left[\frac{1}{r} + i0\right] = \frac{1}{re^{i\theta}} = \frac{1}{z}$.

Properties of Logarithms

1- $e^{logz} = z , z \neq 0.$

2- $loge^{z} = \ln|e^{z}| + iarge^{z} = x + i(y + 2n\pi) = z + 2ni\pi , n = 0, \pm1, \ldots$

3- $lo\, g(z_1 z_2) = logz_1 + log$

4- $log\frac{z_1}{z_2} = logz_1 - logz_2$, $z_1 \neq z_2 \neq 0$

5- $logz^{\frac{1}{n}} = \frac{1}{n} logz$, $n = 0, \pm1, \ldots$

6- $logz^{n} \neq n. logz$

Proof of 5:

$z = re^{i\emptyset}$, $\emptyset = Argz$

Let n be a positive integer

$$\therefore logz\left(z^{\frac{1}{n}}\right) = \log\{\sqrt[n]{r} . \exp\left(\frac{i(\emptyset + 2k\pi)}{n}\right)\}$$

$$= \ln \sqrt[n]{r} + i\left(\frac{i(\emptyset + 2k\pi)}{n} + 2p\pi\right) = \frac{1}{n}\ln r + i(\frac{\emptyset + 2(pn + k)\pi}{n})$$

$$\therefore log\, z^{\frac{1}{n}} = \frac{1}{n}\{\ln r + i(\emptyset + 2q\pi)\} \quad , \qquad n = 1,2,3, \ldots$$

$$= \frac{1}{n}\ logz.$$

Ex. for (6)

Let $z = i$ $, n = 2$,,,, $\log(i)^2 \neq 2\, logi$?

L.H.S. $\log(i)^2 \log(-1) = \ln 1 + i(\pi + 2k\pi) = i\pi(1 + 2k)$

R.H.S. $2logi = 2[\ln 1 + i(\frac{\pi}{2} + 2k\pi)] = i\pi(1 + 4k)$

$\therefore \log(i)^2 \neq 2\, logi$.

Note: when $k = 0$, it mean that we get the equality only in case of the principle value : $\text{Log}(i)^2 = 2\, Logi$.

Ex. Solve the eq. $e^{2z-1} = 1 - i$?

Sol. $1 - i = \sqrt{2}\left[\cos\left(\frac{7\pi}{4} + 2n\pi\right) + isin\left(\frac{7\pi}{4} + 2n\pi\right)\right] = e^{ln\sqrt{2}} + i\left(\frac{7\pi}{4} + 2n\pi\right)$

$\therefore 2z - 1 = \frac{1}{2} ln2 + i\left(\frac{7\pi}{4} + 2n\pi\right)$

$\therefore z = \frac{1}{2} + \frac{1}{4} ln2 + i\left(\frac{7\pi}{8} + n\pi\right)$ $, n = 0, \pm 1, \pm 2 \dots$.