



Foundation of Mathematics 2

CHAPTER 2 SYSTEM OF NUMBERS

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Chapter Two

System of Numbers

1. Natural Numbers

Let $0 = \text{Set with no point, that is; } 0 = \emptyset$, $1 = \text{Set with one point, that is; } 1 = \{0\}$,
 $2 = \text{Set with two points, that is; } 2 = \{0,1\}$, and so on. Therefore,

$$1 = \{0\} = \{\emptyset\},$$

$$2 = \{0,1\} = \{\emptyset, \{\emptyset\}\},$$

$$3 = \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\},$$

$$4 = \{0,1,2,3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\},$$

\vdots

$$n = \{0,1,2,3, \dots, n-1\}.$$

Definition 2.1.1. Let A be a set. A **successor** to A is $A^+ = A \cup \{A\}$ and denoted by A^+ .

According to above definition we can get the numbers $0,1,2,3, \dots$ as follows:

$$0 = \emptyset,$$

$$1 = \{0\} = \emptyset \cup \{\emptyset\} = \emptyset^+ = 0^+,$$

$$2 = \{0,1\} = \{0\} \cup \{1\} = 1 \cup \{1\} = 1^+,$$

$$3 = \{0,1,2\} = \{0,1\} \cup \{2\} = 2 \cup \{2\} = 2^+,$$

Definition 2.1.2. A set A is said to be **successor set** if it satisfies the following conditions:

(i) $\emptyset \in A$,

(ii) if $a \in A$, then $a^+ \in A$.

Remark 2.1.3.

- (i) Any successor set should contains the numbers $0,1,2, \dots n$.
- (ii) Collection of all successor sets is not empty.
- (iii) Intersection of any non-empty collection of successor sets is also successor set.

Definition 2.1.4. Intersection of all successor sets is called **the set of natural numbers** and denoted by \mathbb{N} , and each element of \mathbb{N} is called **natural element**.

Peano’s Postulate 2.1.5.

- (P₁) $0 \in \mathbb{N}$.
- (P₂) If $a \in \mathbb{N}$, then $a^+ \in \mathbb{N}$.
- (P₃) $0 \neq a^+ \in \mathbb{N}$ for every natural number a .
- (P₄) If $a^+ = b^+$, then $a = b$ for any natural numbers a, b .
- (P₅) If X is a successor subset of \mathbb{N} , then $X = \mathbb{N}$.

Remark 2.1.6.

- (i) P₁ says that 0 should be a natural number.
- (ii) P₂ states that the relation $+: \mathbb{N} \rightarrow \mathbb{N}$, defined by $+(n) = n^+$ is mapping.
- (iii) P₃ as saying that 0 is the first natural number, or that ‘- 1’ is not an element of \mathbb{N} .
- (iv) P₄ states that the map $+: \mathbb{N} \rightarrow \mathbb{N}$ is injective.
- (v) P₅ is called the **Principle of Induction**.

2.1.7. Addition + on \mathbb{N}

We will now define the operation of addition $+$ using only the information provided in the Peano’s Postulates.

Let $a, b \in \mathbb{N}$. We define $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ as follows:

$$+(a, b) = a + b = \begin{cases} a + 0 = a & \text{if } b = 0 \\ a + c^+ = (a + c)^+ & \text{if } b \neq 0 \end{cases}$$

where $b = c^+$.

Therefore, if we want to compute $1 + 1$, we note that $1 = 0^+$ and get

$$1 + 1 = 1 + 0^+ = (1 + 0)^+ = 1^+ = 2.$$

We can proceed further to compute $1 + 2$.

To do so, we note that $2 = 1^+$ and therefore that

$$1 + 2 = 1 + 1^+ = (1 + 1)^+ = 2^+ = 3.$$

2.1.8. Multiplication \cdot on \mathbb{N}

We will now define the operation of multiplication \cdot using only the information provided in the Peano's Postulates.

Let $a, b \in \mathbb{N}$. We define $\cdot: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ as follows:

$$\cdot(a, b) = a \cdot b = \begin{cases} a \cdot 0 = 0 & \text{if } b = 0 \\ a \cdot c^+ = a + a \cdot c & \text{if } b \neq 0 \end{cases}$$

where $b = c^+$.

Thus, we can easily show that $a \cdot 1 = a$ by noting that $1 = 0^+$ and therefore,

$$a \cdot 1 = a \cdot 0^+ = a + (a \cdot 0) = a + 0 = a.$$

We can use this to multiply $3 \cdot 2$. Of course, we know that $2 = 1^+$ and therefore,

$$3 \cdot 2 = 3 \cdot 1^+ = 3 + (3 \cdot 1) = 3 + 3 = 3 + 2^+ = (3 + 2)^+ = 5^+ = 6.$$

Remark 2.1.9. From 2.1.7 and 2.1.8 we can deduce that for all $n \in \mathbb{N}$, if $n \neq 0$, then there exist an element $m \in \mathbb{N}$ such that $n = m^+$.

Theorem 2.1.10.

(i) $n^+ = n + 1$, $n^+ = 1 + n$, $n = n \cdot 1$, $n = 1 \cdot n$, $0 \cdot n = 0$, $0 + n = n$
 $\forall n \in \mathbb{N}$.

(ii) (Associative property of $+$): $(n + m) + c = n + (m + c)$, $\forall n, m, c \in \mathbb{N}$.

(iii) (Commutative property of $+$): $n + m = m + n$, $\forall n, m \in \mathbb{N}$.

(iv) (Distributive property of \cdot on $+$): $\forall n, m, c \in \mathbb{N}$,

$$\text{From right } (n + m) \cdot c = n \cdot c + m \cdot c,$$

$$\text{From left } c \cdot (n + m) = c \cdot n + c \cdot m \quad (\text{The prove depend on (vi)}).$$

(v) (Commutative property of \cdot): $n \cdot m = m \cdot n$, $\forall n, m \in \mathbb{N}$.

(vi) (Associative property of \cdot): $(n \cdot m) \cdot c = n \cdot (m \cdot c)$, $\forall n, m, c \in \mathbb{N}$.

(vii) The addition operation $+$ defined on \mathbb{N} is unique.

(viii) The multiplication operation \cdot defined on \mathbb{N} is unique.

(ix) (Cancellation Law for $+$): $m + c = n + c$, for some $c \in \mathbb{N} \Leftrightarrow m = n$.

(x) 0 is the unique element such that $0 + m = m + 0 = m$, $\forall m \in \mathbb{N}$.

(xi) 1 is the unique element such that $1 \cdot m = m \cdot 1 = m$, $\forall m \in \mathbb{N}$.

Proof:

$$\begin{aligned} \text{(i) } n^+ &= (n + 0)^+ && (\text{Since } n = n + 0) \\ &= n + 0^+ && (\text{Def. of } +) \\ &= n + 1 && (\text{Since } 0^+ = 1) \end{aligned}$$

(ii) Let $L_{mn} = \{c \in \mathbb{N} | (m+n) + c = m + (n+c)\}$, $m, n \in \mathbb{N}$.

(1) $(m+n) + 0 = m+n = m + (n+0)$; that is, $0 \in L_{mn}$. Therefore, $L_{mn} \neq \emptyset$.

(2) Let $c \in L_{mn}$; that is, $(m+n) + c = m + (n+c)$. To prove $c^+ \in L_{mn}$.

$$\begin{aligned} (m+n) + c^+ &= ((m+n) + c)^+ \\ &= (m + (n+c))^+ \quad (\text{since } c \in L_{mn}) \\ &= m + (n+c)^+ \quad (\text{Def. of } +) \\ &= m + (n+c^+) \quad (\text{Def. of } +) \end{aligned}$$

Thus, $c^+ \in L_{mn}$. Therefore, L_{mn} is a successor subset of \mathbb{N} . So, we get by \mathbf{P}_5 $L_{mn} = \mathbb{N}$.

(iii) Suppose that $L_m = \{n \in \mathbb{N} | m+n = n+m\}$, $m \in \mathbb{N}$. Then prove that L_m is successor subset of \mathbb{N} .

(iv) Suppose that $L_{mn} = \{c \in \mathbb{N} | c \cdot (m+n) = c \cdot m + c \cdot n\}$, $m, n \in \mathbb{N}$. Then prove that L_{mn} is successor subset of \mathbb{N} .

(v) Suppose that $L_m = \{n \in \mathbb{N} | m \cdot n = n \cdot m\}$, $m \in \mathbb{N}$. Then prove that L_m is successor subset of \mathbb{N} .

(vi) Suppose that $L_{mn} = \{c \in \mathbb{N} | (m \cdot n) \cdot c = m \cdot (n \cdot c)\}$, $m, n \in \mathbb{N}$. Then prove that L_{mn} is successor subset of \mathbb{N} .

(vii) Let \oplus be another operation on \mathbb{N} such that

$$\oplus(a, b) = \begin{cases} a \oplus 0 = a & \text{if } b = 0 \\ a \oplus c^+ = (a \oplus c)^+ & \text{if } b \neq 0 \end{cases}$$

where $b = c^+$.

Let $L = \{m \in \mathbb{N} | n + m = n \oplus m, \forall n \in \mathbb{N}\}$.

(1) To prove $0 \in L$.

$n + 0 = n = n \oplus 0$. Thus, $0 \in L$.

(2) To prove that $k^+ \in L$ for every $k \in L$. Suppose $k \in L$.

$$\begin{aligned} n + k^+ &= (n + k)^+ && \text{Def. of } + \\ &= (n \oplus k)^+ && (\text{Since } k \in L) \\ &= n \oplus k^+ && \text{Def. of } \oplus \end{aligned}$$

Thus, $k^+ \in L$.

From (1), (2) we get that L is a successor set and $L \subseteq \mathbb{N}$. From \mathbf{P}_5 we get that $L = \mathbb{N}$.

(viii) Exercise.

(ix) Suppose that

$L = \{c \in \mathbb{N} | m + c = n + c, \text{ for some } c \in \mathbb{N} \Leftrightarrow m = n\}$, $m, n \in \mathbb{N}$. Then prove that L is successor subset of \mathbb{N} .

(x), (xi) Exercise.

Definition 2.1.11. Let $x, y \in \mathbb{N}$. We say that x less than y and denoted by $x < y$ iff there exist $k \neq 0 \in \mathbb{N}$ such that $x + k = y$.

Theorem 2.1.12.

(i) The relation $<$ is transitive relation on \mathbb{N} .

(ii) $0 < n^+$ and $n < n^+$ for all $n \in \mathbb{N}$.

(iii) $0 < m$ or $m = 0$, for all $m \in \mathbb{N}$.

Proof:

(i),(ii),(iii) Exercise.

Theorem 2.1.13. (Trichotomy)

For each $m, n \in \mathbb{N}$ one and only one of the following is true:

(1) $m < n$ or (2) $n < m$ or (3) $m = n$.

Proof:

Let $m \in \mathbb{N}$ and

$L_1 = \{n \in \mathbb{N} | n < m\}$,

$L_2 = \{n \in \mathbb{N} | m < n\}$,

$L_3 = \{n \in \mathbb{N} | n = m\}$,

$M = L_1 \cup L_2 \cup L_3$.

(1) $L_i \neq \emptyset$ and $L_i \subseteq \mathbb{N}$, $i = 1, 2, 3$. Therefore, $M \subseteq \mathbb{N}$ and $M \neq \emptyset$.

(2) To prove that M is a successor set.

(i) To prove that $0 \in M$.

(a) If $m = 0$, then $0 \in L_3 \rightarrow 0 \in M$ (Def. of \cup)

(b) If $m \neq 0$, then $\exists k \in \mathbb{N} \ni$

$m = k^+$

$\rightarrow 0 < k^+ = m$ (Theorem 2.1.12(ii)).

$\rightarrow 0 \in L_1 \rightarrow 0 \in M$

Or

If $m \neq 0$, then $0 < m$ (Theorem 2.1.12(iii)).

$\rightarrow 0 \in L_1 \rightarrow 0 \in M$

(ii) Suppose that $k \in M$. To prove that $k^+ \in M$.

Since $k \in M$, then $k \in L_1$ or $k \in L_2$ or $k \in L_3$ (Def. of \cup)

(a) If $k \in L_1$

$\rightarrow k < m$ (Def. of L_1)

$\rightarrow \exists c \neq 0 \in \mathbb{N} \ni m = k + c$ (Def of $<$)

$\rightarrow \exists l \in \mathbb{N} \ni c = l^+$ (Remark 2.1.9)

- $\rightarrow m = k + c = k + l^+$ (Def. of +)
 $\quad \quad \quad = (k + l)^+$
 $\rightarrow m = (k + l)^+ = (l + k)^+$ (Commutative law for +)
 $\rightarrow m = l + k^+$ (Def. of +)
 - If $l = 0$, then $m = k^+ \rightarrow k^+ \in L_3$;
 - If $l \neq 0$, then $k^+ < m$ (Def. of $<$) $\rightarrow k^+ \in L_1$.

- (b) If $k \in L_2$
- $\rightarrow m < k$ (Def. of L_2)
 $\rightarrow m < k < k^+$ (Theorem 2.1.12(ii))
 $\rightarrow m < k^+$ (Theorem 2.1.12(i))
 $\rightarrow k^+ \in L_2$ (Def. of L_2)
 $\rightarrow k^+ \in M$ (Def. of U)

- (c) If $k \in L_3$
- $\rightarrow m = k$ (Def. of L_2)
 $\rightarrow m = k < k^+$ (Theorem 2.1.12(ii))
 $\rightarrow m < k^+$ (Theorem 2.1.12(i))
 $\rightarrow k^+ \in L_2$ (Def. of L_2)
 $\rightarrow k^+ \in M$ (Def. of U)

Theorem 2.1.14.

- (i) For all $n \in \mathbb{N}$, $0 < n \Leftrightarrow n \neq 0$.
 (ii) For all $m, n \in \mathbb{N}$, if $n \neq 0$, then $m + n \neq 0$.
 (iii) $m + k < n + k \Leftrightarrow m < n$, for all $m, n, k \in \mathbb{N}$.
 (iv) If $m \cdot n = 0$, then either $m = 0$ or $n = 0$, $\forall m, n \in \mathbb{N}$. (\mathbb{N} has no zero divisor)
 (v) (**Cancellation Law for \cdot**): $m \cdot c = n \cdot c$, for some $c (\neq 0) \in \mathbb{N} \Leftrightarrow m = n$.
 (vi) For all $k (\neq 0) \in \mathbb{N}$, if $m < n$, then $m \cdot k < n \cdot k$, for all $m, n \in \mathbb{N}$.
 (vii) For all $k (\neq 0) \in \mathbb{N}$, if $m \cdot k < n \cdot k$, then $m < n$, for all $m, n \in \mathbb{N}$.

Proof:

(ii) Case 1:

- If $m = 0$.
- $\rightarrow m + n = 0 + n = n \neq 0$
 $\rightarrow m + n \neq 0$

Case 2:

- If $m \neq 0 \rightarrow 0 < m$ By (i)
 Suppose that $m + n = 0$

$\rightarrow m < 0$

$\rightarrow m < 0$ and $0 < m$

Contradiction with Trichotomy Theorem; that is , $m + n \neq 0$.

(vii) Let $m \cdot k < n \cdot k$. Assume that $m \not< n$

$\rightarrow n < m$ or $n = m$ (Trichotomy Theorem)

Suppose $n = m$

$\rightarrow m \cdot k = n \cdot k$ (Cancelation law of \cdot)

$\rightarrow m \cdot k = n \cdot k$ and $m \cdot k < n \cdot k$

\rightarrow Contradiction with (Trichotomy Theorem)

Suppose $n < m$

$\rightarrow n \cdot k < m \cdot k$ (From (vi))

$\rightarrow n \cdot k < m \cdot k$ and $m \cdot k < n \cdot k$

\rightarrow Contradiction with Trichotomy Theorem

$\rightarrow \therefore m < n$

(i),(iii),(iv),(v),(vi) Exercise.

2. Construction of Integer Numbers

Let write $\mathbb{N} \times \mathbb{N}$ as follows:

$$\mathbb{N} \times \mathbb{N} = \left\{ \begin{array}{cccccc} (0,0) & (0,1) & (0,2) & (0,3) & (0,4) & \cdots & \cdots & \cdots & \cdots \\ (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & \cdots & \cdots & \cdots & \cdots \\ (2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \cdots & \cdots & \cdots & \cdots \\ (3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots & \cdots & \cdots & \cdots \\ (4,0) & (4,1) & (4,2) & (4,3) & (4,4) & \cdots & \cdots & \cdots & \cdots \\ (5,0) & (5,1) & (5,2) & (5,3) & (5,4) & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right\}$$

Let define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$\boxed{(a, b)R^*(c, d) \Leftrightarrow a + d = b + c}$$

Example 2.2.1. $(1,0)R^*(4,3)$ since $1 + 3 = 0 + 4$.

$(1,0)R^*(6,4)$ since $1 + 4 \neq 0 + 6$.

Theorem 2.2.2. The relation R^* on $\mathbb{N} \times \mathbb{N}$ is an equivalence relation.

Proof:

(1) Reflexive. For all $(a, b) \in \mathbb{N} \times \mathbb{N}$, $a + b = a + b$; that is $(a, b)R^*(a, b)$.

(2) Symmetric. Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ such that $(a, b)R^*(c, d)$. To prove that $(c, d)R^*(a, b)$.

$$\rightarrow a + d = b + c \quad (\text{Def. of } R^*)$$

$$\rightarrow d + a = c + b \quad (\text{Comm. law for } +)$$

$$\rightarrow c + b = d + a \quad (\text{Equal properties})$$

$$\rightarrow (c, d)R^*(a, b) \quad (\text{Def. of } R^*)$$

(3) Transitive. Let $(a, b), (c, d), (r, s) \in \mathbb{N} \times \mathbb{N}$ such that $(a, b)R^*(c, d)$ and $(c, d)R^*(r, s)$. To prove $(a, b)R^*(r, s)$.

$$a + d = b + c \quad (\text{Since } (a, b)R^*(c, d)) \quad \dots(1)$$

$$c + s = d + r \quad (\text{Since } (c, d)R^*(r, s)) \quad \dots(2)$$

$$\rightarrow (a + d) + s = (b + c) + s \quad (\text{Add } s \text{ to both side of (1)})$$

$$= b + (c + s) \quad (\text{Cancellations low and asso. law for } +) \quad \dots(3)$$

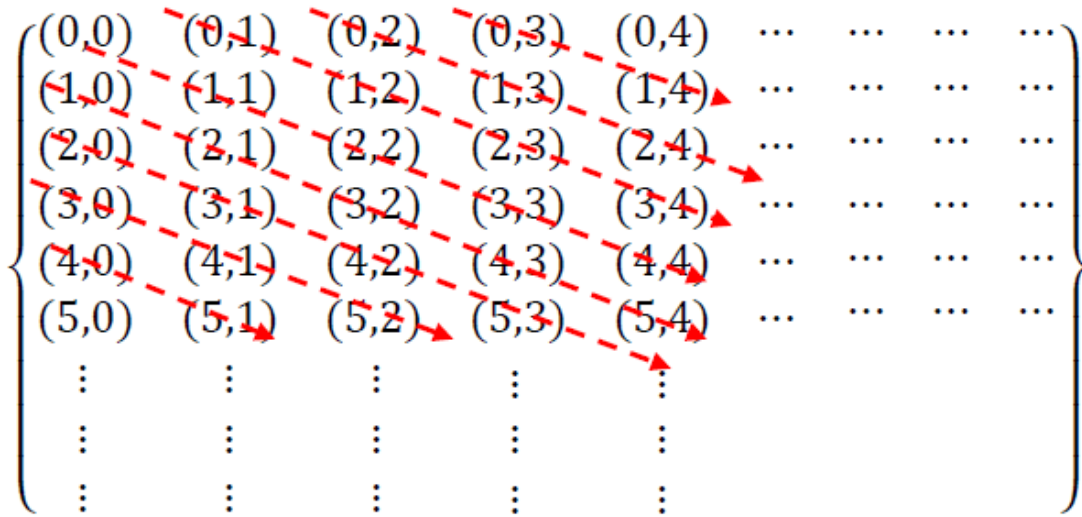
$$\rightarrow (a + d) + s = b + (c + s) \quad (\text{Sub.(2) in (3)})$$

$$\begin{aligned}
 &= b + (d + r) \\
 \rightarrow a + (d + s) &= b + (r + d) \quad (\text{Asso. law and comm. law for } +) \\
 \rightarrow a + (s + d) &= b + (r + d) \quad (\text{Comm. law for } +) \\
 \rightarrow (a + s) + d &= (b + r) + d \quad (\text{Asso. law for } +) \\
 \rightarrow (a + s) &= (b + r) \quad (\text{Cancellation low for } +) \\
 \rightarrow (a, b)R^*(r, s) & \quad (\text{Def. of } R^*)
 \end{aligned}$$

Remark 2.2.3.

(i) The equivalence class of each $(a, b) \in \mathbb{N} \times \mathbb{N}$ is as follows:

$$[(a, b)] = [a, b] = \{(r, s) \in \mathbb{N} \times \mathbb{N} \mid a + s = b + r\}.$$



$$\begin{aligned}
 [1,0] &= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid 1 + y = 0 + x\} \\
 &= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x = 1 + y\} \\
 &= \{(y + 1, y) \mid y \in \mathbb{N}\} \\
 &= \{(1,0), (2,1), (3,2), \dots\}.
 \end{aligned}$$

$$\begin{aligned}
 [0,0] &= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid 0 + y = 0 + x\} \\
 &= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x = y\} \\
 &= \{(x, x) \mid x \in \mathbb{N}\} \\
 &= \{(0,0), (1,1), (2,2), \dots\}.
 \end{aligned}$$

(ii) $[a, b] = \{(a, b), (a + 1, b + 1), (a + 2, b + 2), \dots\}.$

(iii) These classes $[(a, b)]$ formed a partition on $\mathbb{N} \times \mathbb{N}$.

Theorem 2.2.4. For all $(x, y) \in \mathbb{N} \times \mathbb{N}$, one of the following hold:

- (i) $[x, y] = [0,0]$, if $x = y$.
- (ii) $[x, y] = [z, 0]$, for some $z \in \mathbb{N}$, if $y < x$.

(iii) $[x, y] = [0, z]$, for some $z \in \mathbb{N}$, if $x < y$.

Proof:

Let $(x, y) \in \mathbb{N} \times \mathbb{N}$. Then by Trichotomy Theorem, there are three possibilities.

(1) $x = y$,

$$\rightarrow 0 + y = 0 + x$$

Def. of +

$$\rightarrow (0,0)R^*(x, y)$$

Def. of R^*

$$\rightarrow [0,0] = [x, y]$$

Def. of $[a, b]$

(2) $x < y$,

$$\rightarrow y = x + z \text{ for some } z \in \mathbb{N}$$

Def. of $<$

$$\rightarrow x + z = y + 0$$

Def. of +

$$\rightarrow (x, y)R^*(0, z) \rightarrow (0, z)R^*(x, y)$$

Def. of R^*

$$\rightarrow [0, z] = [x, y]$$

Def. of $[a, b]$

(3) $y < x$,

$$\rightarrow x = y + z \text{ for some } z \in \mathbb{N}$$

Def. of $<$

$$\rightarrow x + 0 = y + z$$

Def. of +

$$\rightarrow (x, y)R^*(z, 0) \rightarrow (z, 0)R^*(x, y)$$

Def. of R^*

$$\rightarrow [z, 0] = [x, y]$$

Def. of $[a, b]$

2.2.5. Constriction of Integer Numbers \mathbb{Z}

The set of integer numbers, \mathbb{Z} will be defined as follows:

$$\mathbb{Z} = \bigcup_{(a,b) \in \mathbb{N} \times \mathbb{N}} [(a, b)] = \bigcup_{a(\neq 0) \in \mathbb{N}} [(a, 0)] \bigcup_{b(\neq 0) \in \mathbb{N}} [(0, b)] \bigcup [(0,0)].$$

2.2.6. Addition, Subtraction and Multiplication on \mathbb{Z}

Addition: $\oplus: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$;

$$\boxed{[r, s] \oplus [t, u] = [r + t, s + u]}$$

Subtraction: $\ominus: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$;

$$\boxed{[r, s] \ominus [t, u] = [r, s] \oplus [u, t] = [r + u, s + t]}$$

Multiplication: $\odot: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$;

$$\boxed{[r, s] \odot [t, u] = [r \cdot t + s \cdot u, r \cdot u + s \cdot t]}$$

Theorem 2.2.7. The relations \oplus , \ominus and \odot are well defined; that is, \oplus , \ominus and \odot are functions.

Proof:

To prove \oplus is function. Assume that $[r, s] = [r_0, s_0]$ and $[t, u] = [t_0, u_0]$.

$$[r, s] \oplus [t, u] = [r + t, s + u]$$

$$[r_0, s_0] \oplus [t_0, u_0] = [r_0 + t_0, s_0 + u_0]$$

To prove $[r + t, s + u] = [r_0 + t_0, s_0 + u_0]$.

$$\rightarrow (r, s)R^*(r_0, s_0)$$

$$[r, s] = [r_0, s_0] \text{ and Def. of } R^*$$

$$\rightarrow r + s_0 = s + r_0$$

$$\dots\dots(1)$$

$$\rightarrow (t, u)R^*(t_0, u_0)$$

$$[r, s] = [r_0, s_0] \text{ and Def. of } R^*$$

$$\rightarrow t + u_0 = u + t_0$$

$$\dots\dots(2)$$

$$\rightarrow (r + s_0) + (t + u_0) = (s + r_0) + (u + t_0)$$

Adding (1), (2)

$$\rightarrow (r + t) + (s_0 + u_0) = (s + u) + (r_0 + t_0)$$

Asso. and comm. for +

$$\rightarrow (r + t, s + u)R^*(r_0 + t_0, s_0 + u_0)$$

Def. of R^*

$$\rightarrow [r + t, s + u] = [r_0 + t_0, s_0 + u_0]$$

Def. of $[a, b]$

\ominus and \odot (Exercise)

Example 2.2.8.

$$[2, 4] \oplus [0, 1] = [2 + 0, 4 + 1] = [2, 5] = [0, 3].$$

$$[5, 2] \oplus [8, 1] = [5 + 8, 2 + 1] = [13, 3] = [10, 0].$$

Notation 2.2.9.

(i) Let identify the equivalence classes $[r, s]$ according to its form as in Theorem 2.2.3.

$$[a, 0] = +a, a \in \mathbb{N}, \text{ called } \mathbf{positive\ integer}.$$

$$[0, b] = -b, b \in \mathbb{N}, \text{ called } \mathbf{negative\ integer}.$$

$$[0, 0] = 0, \text{ called the } \mathbf{zero\ element}.$$

$$[4, 6] = [0, 2] = -2$$

$$[9, 6] = [3, 0] = 3$$

$$[6, 6] = [0, 0] = 0$$

(ii) The relation $i: \mathbb{N} \rightarrow \mathbb{Z}$, defined by $i(n) = [n, 0]$ is 1-1 function, and

$i(n + m) = i(n) \oplus i(m)$, $i(n \cdot m) = i(n) \odot i(m)$. So, we can identify n with $+n$;

that is, $\boxed{+n = n}$, $\boxed{+ = \oplus}$ and $\boxed{\cdot = \odot}$.

Theorem 2.2.10.

- (i) $a \in \mathbb{Z}$ is positive if there exist $[x, y] \in \mathbb{Z}$ such that $a = [x, y]$ and $y < x$.
(ii) $b \in \mathbb{Z}$ is negative if there exist $[x, y] \in \mathbb{Z}$ such that $b = [x, y]$ and $x < y$.
(iii) For each element $[x, y] \in \mathbb{Z}$, $[y, x] \in \mathbb{Z}$ is the unique element such that $[x, y] + [y, x] = 0$. Denote $[y, x]$ by $-[x, y]$.
(iv) $\boxed{(-m) \odot n = -(m \cdot n)}$, $\forall n, m \in \mathbb{Z}$.
(v) $\boxed{m \odot (-n) = -(m \cdot n)}$, $\forall n, m \in \mathbb{Z}$.
(vi) $\boxed{(-m) \odot (-n) = m \cdot n}$, $\forall n, m \in \mathbb{Z}$.
(vii) (Commutative property of +): $\boxed{n + m = m + n}$, $\forall n, m \in \mathbb{Z}$.
(viii) (Associative property of +): $\boxed{(n + m) + c = n + (m + c)}$, $\forall n, m, c \in \mathbb{Z}$.
(ix) (Commutative property of \cdot): $\boxed{n \cdot m = m \cdot n}$, $\forall n, m \in \mathbb{Z}$.
(x) (Associative property of \cdot): $\boxed{(n \cdot m) \cdot c = n \cdot (m \cdot c)}$, $\forall n, m, c \in \mathbb{Z}$.
(xi) (Cancellation Law for +): $m + c = n + c$, for some $c \in \mathbb{Z} \Leftrightarrow m = n$.
(xii) (Cancellation Law for \cdot): $m \cdot c = n \cdot c$, for some $c (\neq 0) \in \mathbb{Z} \Leftrightarrow m = n$.
(xiii) 0 is the unique element such that $0 + m = m + 0 = m$, $\forall m \in \mathbb{Z}$.
(xiv) 1 is the unique element such that $1 \cdot m = m \cdot 1 = m$, $\forall m \in \mathbb{Z}$.

(xv) Let $a, b, c \in \mathbb{Z}$. Then $\boxed{c = a - b} \Leftrightarrow \boxed{a = c + b}$.

(xvi) $\boxed{-(-b) = b}$, $\forall b \in \mathbb{Z}$.

Proof: Exercise.

Remark 2.2.11.

For each element $a = [x, y] \in \mathbb{Z}$, the unique element in Theorem 2.2.8(xiv) is $-a = [y, x]$.

Definition 2.2.12. (\mathbb{Z} as an Ordered)

Let $[r, s], [t, u] \in \mathbb{Z}$. We say that $[r, s]$ less than $[t, u]$ and denoted by

$$[r, s] < [t, u] \Leftrightarrow r + u < s + t.$$

This is well defined and agrees with the ordering on \mathbb{N} .

Theorem 2.2.13.(Trichotomy For \mathbb{Z}) (Well Ordering)

For each $[r, s], [t, u] \in \mathbb{Z}$ one and only one of the following is true:

- (1) $[r, s] < [t, u]$ or (2) $[t, u] < [r, s]$ or (3) $[r, s] = [t, u]$.

Proof:

Since $r + u, t + s \in \mathbb{N}$, so by Trichotomy Theorem for \mathbb{N} , one and only one of the following is true:

- (1) $r + u < s + t \rightarrow [r, s] < [t, u]$
- (2) $s + t < r + u \rightarrow [t, u] < [r, s]$
- (3) $r + u = s + t \rightarrow (r, s)R^*(t, u) \rightarrow [r, s] = [t, u]$.

Theorem 2.2.14.

For each $[r, s] \in \mathbb{Z}$, $[r, s] < [0, 0] \Leftrightarrow r < s$.

Proof:

$$[r, s] < [0, 0] \Leftrightarrow r + 0 < s + 0 \Leftrightarrow r < s.$$

Remark 2.2.15.

According to Theorem 2.2.11 and Notation 2.2.7(i), for all $[r, s] \in \mathbb{Z}$

$$\begin{aligned} [r, s] < [0, 0] &\Leftrightarrow r < s \Leftrightarrow [r, r + l] \in \mathbb{Z}, \text{ where } s = r + l \text{ for some } l \\ &\Leftrightarrow [0, l] < [0, 0] \\ &\Leftrightarrow -l < 0. \end{aligned}$$