8) Let $(\mathbb{R}, \tau_{ray})$ be the ray topological space and let $A = [-1, 10]$, $B = \mathbb{Q}$ & $C = \{-5, 1, 9\}$. Find $\bar{X}$, X°, $b(X)$, $Ext(X)$ & $d(X)$ where $X = A, B$ & C (ch.)

9) Let $\mathbb{N}$ be the set of all natural numbers and let $\tau = \{\phi\} \cup \{A_n \mid n \in \mathbb{N}\}$, where $A_n = \{n, n+1, n+2, \ldots\}$ and let $B = \{10, 11, 12\}$ & $C = \{3, 6, 9, 12, \ldots\}$. Find $\bar{X}$, X°, $b(X)$, $Ext(X)$ & $d(X)$, where $X = B$ & C (ch.)

Remarks: Let (X, τ) be a topological space and $A \subseteq X$. Then:

1. The closure of A is the smallest closed subset of X which contains A
2. The interior of A is the largest open subset of X which contained in A

proof

1. $\bar{A} = \bigcap_{\alpha \in \Lambda} \{F_\alpha \mid F_\alpha \underset{closed}{\subseteq} X$ & $A \subseteq F_\alpha\}$. since $F_\alpha \underset{closed}{\subseteq} X \; \forall \alpha \in \Lambda$

$\xrightarrow{\text{By Th(2)}} \bigcap_{\alpha \in \Lambda} F_\alpha \underset{closed}{\subseteq} X \Longrightarrow \bar{A} \underset{closed}{\subseteq} X$

$\therefore A \subseteq F_\alpha \; \forall \alpha \in \Lambda \Longrightarrow A \subseteq \bigcap_{\alpha \in \Lambda} F_\alpha \Longrightarrow A \subseteq \bar{A}$

Now, let $F \underset{closed}{\subseteq} X$ & $A \subseteq F$

$\Longrightarrow \exists \alpha \in \Lambda$ s.t $F_\alpha = F$

$\therefore \bigcap_{\alpha \in \Lambda} F_\alpha \subseteq F_\alpha \; \forall \alpha \in \Lambda \Longrightarrow \bigcap_{\alpha \in \Lambda} F_\alpha \subseteq F \Longrightarrow \bar{A} \subseteq F$

$\therefore \bar{A}$ is the smallest closed subset of X which contains A

2. $A^\circ = \bigcup_{\alpha \in \Lambda} \{U_\alpha \mid U_\alpha \underset{open}{\subseteq} X$ & $U_\alpha \subseteq A\}$

$\therefore U_\alpha \underset{open}{\subseteq} X \; \forall \alpha \in \Lambda \xrightarrow[\text{of top}]{\text{by def}} \bigcup_{\alpha \in \Lambda} U_\alpha \underset{open}{\subseteq} X \Longrightarrow A^\circ \underset{open}{\subseteq} X$

$\because U_\alpha \subseteq A \ \forall \alpha \in \Lambda \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \subseteq A \Rightarrow A^\circ \subseteq A$

Now, let $U \underset{open}{\subseteq} X$ & $U \subseteq A$

$\Rightarrow \exists \alpha \in \Lambda$ s.t $U_\alpha = U$

$\because U_\alpha \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \ \forall \alpha \in \Lambda \Rightarrow U \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \Rightarrow U \subseteq A^\circ$

$\therefore A^\circ$ is the largest open subset of X which contained in A.

Theorem (3): Let (X, τ) be a topological space and let A and B are subsets of X, then:

① If $A \subseteq B \Rightarrow d(A) \subseteq d(B)$

② $d(A \cup B) = d(A) \cup d(B)$

③ $d(A \cap B) \subseteq d(A) \cap d(B)$ (ch.)

④ A is closed subset of X iff $d(A) \subseteq A$

⑤ $\bar{A} = A \cup d(A)$

⑥ If $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$ ⑦ If $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$

proof

① Let $A \subseteq B$ T.p $d(A) \subseteq d(B)$

Let $x \in d(A)$ T.p $x \in d(B)$

Let $U \in \tau$ & $x \in U \xrightarrow{x \in d(A)} (U - \{x\}) \cap A \neq \phi$

$\because A \subseteq B \Rightarrow \phi \neq (U - \{x\}) \cap A \subseteq (U - \{x\}) \cap B \Rightarrow (U - \{x\}) \cap B \neq \phi$

$\Rightarrow x \in d(B) \Rightarrow d(A) \subseteq d(B)$

---

② $\because A \subseteq A \cup B \xrightarrow{by ①} d(A) \subseteq d(A \cup B)$

$\because B \subseteq A \cup B \xrightarrow{by ①} d(B) \subseteq d(A \cup B)$

$\Rightarrow d(A) \cup d(B) \subseteq d(A \cup B) \longrightarrow ①$

T.p $d(A \cup B) \subseteq d(A) \cup d(B)$

Let $x \in d(A \cup B) \Rightarrow \forall U \in \tau$ & $x \in U$, $(U - \{x\}) \cap (A \cup B) \neq \phi$

$\Rightarrow [(U - \{x\}) \cap A] \cup [(U - \{x\}) \cap B] \neq \phi$

$\Rightarrow$ either $(U-\{x\})\cap A \neq \phi$ or $(U-\{x\})\cap B \neq \phi$

$\Rightarrow x \in d(A)$ or $x \in d(B) \Rightarrow x \in d(A) \cup d(B)$

$\Rightarrow d(A\cup B) \subseteq d(A) \cup d(B) \rightarrow$ ②

From ① & ② we have $d(A\cup B) = d(A) \cup d(B)$

④ A is closed subset of $X \Leftrightarrow d(A) \subseteq A$

proof $\Rightarrow$ suppose that $A \underset{closed}{\subseteq} X$ T.p $d(A) \subseteq A$

Let $x \notin A \Rightarrow x \in A^c \underset{open}{\subseteq} X$

$\because A^c \cap A = \phi \Rightarrow (A^c - \{x\}) \cap A = \phi \Rightarrow x \notin d(A)$

$\Rightarrow d(A) \subseteq A$

$\Leftarrow$ suppose that $d(A) \subseteq A$ T.p A is closed in X

i.e T.p A^c is open ?

Let $x \in A^c \Rightarrow x \notin A$

$\because d(A) \subseteq A \Rightarrow x \notin d(A) \Rightarrow \exists U_x \underset{open}{\subseteq} X$ s.t $x \in U_x$ &

$(U_x - \{x\}) \cap A = \phi \xrightarrow{x \notin A} U_x \cap A = \phi \Rightarrow U_x \subseteq A^c \quad \forall x \in A^c$

$\Rightarrow \bigcup_{x \in A^c} U_x \subseteq A^c \rightarrow$ ①

$\because A^c = \bigcup_{x\in A^c} \{x\} \subseteq \bigcup_{x \in A^c} U_x \Rightarrow A^c \subseteq \bigcup_{x\in A^c} U_x \rightarrow$ ②

From ① & ② $\Rightarrow A^c = \bigcup_{x \in A^c} U_x \Rightarrow A^c$ is open (A^c = union of open sets)

$\Rightarrow A$ is closed in X

5) $\bar{A} = A \cup d(A)$

proof T.p $\bar{A} \subseteq A \cup d(A)$

suppose that $x \notin A \cup d(A) \Rightarrow x \notin A$ & $x \notin d(A)$

$\because x \notin d(A) \Rightarrow \exists U \in \tau$ s.t $x \in U$ & $(U - \{x\}) \cap A = \phi$

$\xrightarrow{x \notin A} U \cap A = \phi \Rightarrow A \subseteq U^c \underset{closed}{\subseteq} X$

$\because A \subseteq \overline{A}$ & $\overline{A}$ is the smallest closed subset of X which contain $A \longrightarrow \overline{A} \subseteq U^c$

$\because x \in U \longrightarrow x \notin U^c \longrightarrow x \notin \overline{A}$

$\therefore \overline{A} \subseteq A \cup d(A) \longrightarrow$ ①

T.P $A \cup d(A) \subseteq \overline{A}$

Let $x \in A \cup d(A) \longrightarrow x \in A$ or $x \in d(A)$ (or both)

If $x \in A \xrightarrow{A \subseteq \overline{A}} x \in \overline{A} \longrightarrow A \cup d(A) \subseteq \overline{A}$

& if $x \in d(A)$ T.P $x \in \overline{A}$

Suppose that $x \notin \overline{A} \longrightarrow \exists F \underset{closed}{\subseteq} X$ s.t $x \notin F$ & $A \subseteq F$

$\because x \notin F \Longrightarrow x \in F^c \underset{open}{\subseteq} X$

$\because A \subseteq F \longrightarrow F^c \cap A = \phi \longrightarrow (F^c \setminus \{x\}) \cap A = \phi \longrightarrow x \notin d(A)$ C

$\therefore x \in \overline{A} \Longrightarrow A \cup d(A) \subseteq \overline{A} \longrightarrow$ ②

From ① & ② we get $\overline{A} = A \cup d(A)$

⑥ If $A \subseteq B \longrightarrow \overline{A} \subseteq \overline{B}$

proof

$\because A \subseteq B \subseteq \overline{B} \longrightarrow A \subseteq \overline{B}$

$\longrightarrow \overline{B}$ is a closed subset of X s.t $A \subseteq \overline{B}$

$\because A \subseteq \overline{A}$ & $\overline{A}$ is the smallest closed subset of X which contains A

$\longrightarrow \overline{A} \subseteq \overline{B}$

طريقة أخرى

If $A \subseteq B \xrightarrow{by ①} d(A) \subseteq d(B) \longrightarrow A \cup d(A) \subseteq B \cup d(B) \longrightarrow$

$\xrightarrow{by ⑤} \overline{A} \subseteq \overline{B}$

⑦ If $A \subseteq B \longrightarrow A^\circ \subseteq B^\circ$

proof

$\because A^\circ \subseteq A \subseteq B \longrightarrow A^\circ \subseteq B$

$\longrightarrow A^\circ$ is an open subset of X s.t $A^\circ \subseteq B$

$\because B^\circ \subseteq B$ & B° is the largest open subset of X which contained in B

$\longrightarrow A^\circ \subseteq B^\circ$