3) $b(A)=\{x\in\mathbb{R} \mid \forall U\in D \ \& \ x\in U,\ U\cap A\neq\phi \ \& \ U\cap A^c\neq\phi\}$

Let $x\in\mathbb{R}$

$\because \mathbb{R}$ has the discrete topology $\Rightarrow \{x\}\underset{open}{\subseteq}\mathbb{R} \ \& \ x\in\{x\}$.

if $x\in A \Rightarrow \{x\}\cap A^c=\phi \Rightarrow x\notin b(A)$.

if $x\in A^c \Rightarrow \{x\}\cap A=\phi \Rightarrow x\notin b(A)$

$\Rightarrow b(A)=\phi$

$b(B)=\phi \ \& \ b(C)=\phi$ (ch)

4) $Ext(A)=\overline{A}^c=(0,1]^c=\mathbb{R}-(0,1]$

$Ext(B)=\overline{B}^c=\mathbb{Q}^c=\mathbb{R}-\mathbb{Q} \quad \& \ Ext(C)=\overline{C}^c=\{1,4,9\}^c=\mathbb{R}-\{1,4,9\}$

5) $d(A)=\{x\in\mathbb{R} \mid \forall U\in D \ \& \ x\in U,\ (U-\{x\})\cap A\neq\phi\}$

Let $x\in\mathbb{R} \xrightarrow{\tau=D} \{x\}\underset{open}{\subseteq}\mathbb{R} \ \& \ x\in\{x\}$.

$\because (\{x\}-\{x\})\cap A=\phi\cap A=\phi \Rightarrow x\notin d(A) \Rightarrow d(A)=\phi$

$d(B)=d(C)=\phi$ (ch)

6) Let $(\mathbb{R},\mathcal{U})$ be the usual topological space and let $A=[-2,5)$, $B=\mathbb{Q}$ & $C=\{-1,2,0\}$. Find $\overline{X}$, X°, $b(X)$, $Ext(X)$ & $d(X)$, where $X=A,B$ & C

Solution

$\mathcal{U}=\{U \mid U\subseteq\mathbb{R} \ \& \ \forall x\in U, \exists \epsilon>0 \text{ s.t } N_\epsilon(x)\subseteq U\}$

open sets in $\mathbb{R}$ are $\phi, \mathbb{R}, (a,b), (-\infty,a)$ & (a,∞) where $a,b\in\mathbb{R}$

$\mathcal{F}=\{\phi, \mathbb{R}, [a,b], (-\infty,a], [a,\infty), \mathbb{Z}, \mathbb{N}, \text{any finite set}\}$

Let $A=[-2,5)$

1) $\therefore \overline{A}=\cap\{F \mid F\underset{closed}{\subseteq}\mathbb{R} \ \& \ A\subseteq F\}=[-2,5]$

$\overline{B}=\cap\{F \mid F\underset{closed}{\subseteq}\mathbb{R} \ \& \ B\subseteq F\}=\mathbb{R} \quad \& \ \overline{C}=\{-1,2,0\}$

2) $A^\circ=\cup\{U \mid U\underset{open}{\subseteq}\mathbb{R} \ \& \ U\subseteq A\}=(-2,5)$

$B^\circ = \phi$ & $C^\circ = \phi$

③ $b(A) = \{x \in \mathbb{R} \mid \forall U \in \mathcal{U} \,\&\, x \in U,\ U \cap A \neq \phi \,\&\, U \cap A^c \neq \phi\}$

Let $x \in \mathbb{R}$

If $x \in (-2,5) \Rightarrow (-2,5) \underset{open}{\subseteq} \mathbb{R}$ & $x \in (-2,5)$

$\therefore (-2,5) \cap A \neq \phi$ & $(-2,5) \cap A^c = \phi \Rightarrow x \notin b(A)$

If $x < -2 \Rightarrow (-\infty,-2) \underset{open}{\subseteq} \mathbb{R}$ & $x \in (-\infty,-2)$

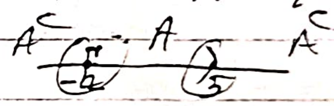
$\therefore (-\infty,-2) \cap A = \phi$ & $(-\infty,-2) \cap A^c \neq \phi \Rightarrow x \notin b(A)$

& If $x > 5 \Rightarrow (5,\infty) \underset{open}{\subseteq} \mathbb{R}$ & $x \in (5,\infty)$

$\therefore (5,\infty) \cap A = \phi$ and $(5,\infty) \cap A^c \neq \phi \Rightarrow x \notin b(A)$

If $x = -2$ & $x = 5 \Rightarrow \forall U \in \mathcal{U}$ & $x \in U$, $U \cap A \neq \phi$ &

$U \cap A^c \neq \phi \Rightarrow x \in b(A)$



$\therefore b(A) = \{-2,5\}$

$b(B) = \{x \in \mathbb{R} \mid \forall U \in \mathcal{U} \,\&\, x \in U,\ U \cap B \neq \phi \,\&\, U \cap B^c \neq \phi\}$

Let $x \in \mathbb{R}$

$\therefore \forall U \in \mathcal{U}$ & $x \in U$, $U \cap \mathbb{Q} \neq \phi$ & $U \cap \mathbb{Q}^c \neq \phi$

(since U contains rational and irrational numbers)

$b(B) = \mathbb{R}$

$b(C) = \{x \in \mathbb{R} \mid \forall U \in \mathcal{U} \,\&\, x \in U,\ U \cap C \neq \phi \,\&\, U \cap C^c \neq \phi\}$

let $x \in \mathbb{R} \Rightarrow$ either $x \in C$ or $x \notin C$

If $x \notin C \Rightarrow x \in C^c \underset{open}{\subseteq} \mathbb{R}$ (since C is a finite set)

closed

$\therefore C \cap C^c = \phi \Rightarrow x \notin b(C)$

If $x \in C \Rightarrow \forall U \in \mathcal{U}$ & $x \in U$, $U \cap C \neq \phi$ & $U \cap C^c \neq \phi$

$\Rightarrow x \in b(C) \Rightarrow b(C) = \{-1,2,0\}$

4) $Ext(A) = \bar{A}^c = [-2,5]^c = \mathbb{R} - [-2,5] = (-\infty,-2) \cup (5,\infty)$

$Ext(B) = \bar{B}^c = \mathbb{R}^c = \phi$ & $Ext(C) = \bar{C}^c = \{-1,2,0\}^c = \mathbb{R} - \{-1,2,0\}$ 18

$A=[-2,5)$

(5) $d(A)=\{x\in\mathbb{R} \mid \forall U\in\mathcal{U} \ \& \ x\in U, (U-\{x\})\cap A\neq\phi\}$

Let $x\in\mathbb{R}$

If $x<-2 \Rightarrow (-\infty,-2) \underset{open}{\subseteq} \mathbb{R} \ \& \ x\in(-\infty,-2)$

$\therefore ((-\infty,-2)-\{x\})\cap A=\phi \Rightarrow x\notin d(A)$

& if $x>5 \Rightarrow (5,\infty) \underset{open}{\subseteq} \mathbb{R} \ \& \ x\in(5,\infty)$

$\therefore ((5,\infty)-\{x\})\cap A=\phi \Rightarrow x\notin d(A)$

& if $-2\leq x\leq 5 \Rightarrow \forall U\in\mathcal{U} \ \& \ x\in U, (U-\{x\})\cap A\neq\phi$

$\Rightarrow x\in d(A) \Rightarrow d(A)=[-2,5]$

$d(B)=d(\mathbb{Q})=\mathbb{R}$, since $\forall U\in\mathcal{U} \ \& \ x\in U$,

$(U-\{x\})\cap B\neq\phi$ (since U contains rational & irrational numbers)

$d(C)=\phi$

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(7) Let $(\mathbb{R},\tau_{cof.})$ be the cofinite topological space and let $A=(3,9)$, $B=\mathbb{Q}$ & $C=\{-1,0,1\}$. Find $\overline{X}, X^\circ, b(X), Ext(X)$ & $d(X)$, where $X=A,B$ & C

Solution $\tau_{cof.}=\{U \mid U\subseteq\mathbb{R} \ \& \ U^c \text{ is finite}\}\cup\{\phi\}$

$\mathcal{F}=\{\text{any finite set}\}\cup\{\mathbb{R}\}$

(1) $\overline{A}=\cap\{F \mid F \underset{closed}{\subseteq} \mathbb{R} \ \& \ A\subseteq F\}=\mathbb{R}$

$\overline{B}=\mathbb{R}$ & $\overline{C}=C=\{-1,0,1\}$

(2) $A^\circ=\cup\{U \mid U \underset{open}{\subseteq} \mathbb{R} \ \& \ U\subseteq A\}=\phi$

$B^\circ=\phi$ & $C^\circ=\phi$

(3) $b(A)=\{x\in\mathbb{R} \mid \forall U\in\tau_{cof.} \ \& \ x\in U, U\cap A\neq\phi \ \& \ U\cap A^c\neq\phi\}$

Let $x\in\mathbb{R}$ & $U \underset{open}{\subseteq} \mathbb{R}$ s.t $x\in U$

T.P $U \cap A \neq \phi$ & $U \cap A^c \neq \phi$

if $U \cap A = \phi \Rightarrow \underbrace{A}_{\text{infinite}} \subseteq \underbrace{U^c}_{\text{finite}}$ C!

$\Rightarrow U \cap A \neq \phi$

& if $U \cap A^c = \phi \Rightarrow A^c \subseteq \underbrace{U^c}_{\text{finite}}$ C!

$\Rightarrow U \cap A^c \neq \phi$

$\therefore U \cap A \neq \phi$ & $U \cap A^c \neq \phi \Rightarrow x \in b(A) \Rightarrow b(A) = \mathbb{R}$

$b(B) = b(\mathbb{Q}) = \mathbb{R}$

$b(C) = \{x \in \mathbb{R} \mid \forall U \in \tau_{cof}$ & $x \in U$, $U \cap C \neq \phi$ & $U \cap C^c \neq \phi\}$

Let $x \in \mathbb{R} \Rightarrow x \in C$ or $x \notin C$

if $x \notin C \Rightarrow x \in C^c \underset{open}{\subseteq} \mathbb{R}$

$\therefore C \cap C^c = \phi \Rightarrow x \notin b(C)$

& if $x \in C \Rightarrow \forall U \underset{open}{\subseteq} \mathbb{R}$ & $x \in U$

$U \cap C \neq \phi$ ($x \in U$ & $x \in C$) & $U \cap C^c \neq \phi$, since

if $U \cap C^c = \phi \Rightarrow C^c \subseteq \underbrace{U^c}_{\text{finite}}$ C!

$\Rightarrow U \cap C^c \neq \phi \Rightarrow x \in b(C) \Rightarrow b(C) = \{-1, 0, 1\} = C$

④ $Ext(A) = \overline{A}^c = \mathbb{R}^c = \phi$ & $Ext(B) = \overline{B}^c = \mathbb{R}^c = \phi$

$Ext(C) = \overline{C}^c = C^c = \mathbb{R} - \{-1, 0, 1\}$

⑤ $d(A) = \{x \in \mathbb{R} \mid \forall U \in \tau_{cof}$ & $x \in U$, $(U - \{x\}) \cap A \neq \phi\}$

Let $x \in \mathbb{R}$ & $U \in \tau_{cof}$ s.t $x \in U$

T.P $(U - \{x\}) \cap A \neq \phi$

if $(U - \{x\}) \cap A = \phi \Rightarrow \underbrace{A}_{\text{infinite}} \subseteq \underbrace{U^c}_{\text{finite}} \cup \underbrace{\{x\}}_{\text{fini.}}$ C!

$\therefore (U - \{x\}) \cap A \neq \phi \Rightarrow x \in d(A) \Rightarrow d(A) = \mathbb{R}$

$d(B) = d(\mathbb{Q}) = \mathbb{R}$ & $d(C) = \phi$ ch.