Derived sets

Def: Let (X, τ) be a topological space and $A \subseteq X$. Then

① The closure of A, denoted by $\bar{A}$ or $cl(A)$ is the intersection of all τ-closed subsets of X which contains A

i.e $\bar{A} = cl(A) = \cap\{F \mid F \underset{closed}{\subseteq} X \ \& \ A \subseteq F\}$

② The interior of A, denoted by A° or $int(A)$ is the union of all τ-open subsets of X which contained in A

i.e $A^{\circ} = int(A) = \cup\{U \mid U \underset{open}{\subseteq} X \ \& \ U \subseteq A\}$

③ The frontier or boundary of A, denoted by $Fr(A)$ or $bd(A)$ or $b(A)$ is $b(A) = \{x \in X \mid \forall U \in \tau \ \& \ x \in U, U$ intersects both A and $A^c\}$

i.e $b(A) = \{x \in X \mid \forall U \in \tau \ \& \ x \in U, U \cap A \neq \phi \ \& \ U \cap A^c \neq \phi\}$

④ The exterior of A, denoted by $Ext(A)$ is the complement of $\bar{A}$ i.e $Ext(A) = X - \bar{A} = \bar{A}^c$

⑤ The derived set (or the set of all limit points) of A, denoted by A' or $d(A)$ is:
$A' = d(A) = \{x \in X \mid \forall U \in \tau, x \in U, (U - \{x\}) \cap A \neq \phi\}$

Examples: ① Let $X = \{1, 2, 3\}$ & $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$
Let $A = \{1, 3\}$ & $\mathcal{F} = \{X, \phi, \{2,3\}, \{1,3\}, \{3\}\}$

(1) closed sets which contains A are X, & $\{1,3\}$
$\therefore \bar{A} = \cap\{F \mid F \underset{closed}{\subseteq} X \ \& A \subseteq F\} = X \cap \{1,3\} = \{1,3\}$

(2) open sets which are contained in A are $\{1\}$ & ϕ

$A^\circ = \cup\{U \mid U \underset{open}{\subseteq} X \;\&\; U \subseteq A\} = \{1\} \cup \{\phi\} = \{1\}$

③ $b(A) = \{x \in X \mid \forall U \in \tau \;\&\; x \in U,\; U \cap A \neq \phi \;\&\; U \cap A^c \neq \phi\}$

$1 \in \{1\} \in \tau \;\&\; A^c = \{2\}$

$\because \{1\} \cap A^c = \{1\} \cap \{2\} = \phi \rightarrow 1 \notin b(A)$

$2 \in \{2\} \in \tau$

$\because \{2\} \cap A = \{2\} \cap \{1,3\} = \phi \Rightarrow 2 \notin b(A)$

$3 \in X \in \tau$

$\because X \cap A = A \neq \phi \;\&\; X \cap A^c = A^c \neq \phi$

$\because X$ is the only open set which contains 3

$\Rightarrow 3 \in b(A) \rightarrow b(A) = \{3\}$

④ $Ext(A) = \overline{A}^c = \{1,3\}^c = \{2\}$

⑤ $d(A) = \{x \in X \mid \forall U \in \tau \;\&\; x \in U,\; (U - \{x\}) \cap A \neq \phi\}$

$\because 1 \in \{1\} \in \tau \;\&\; (\{1\} - \{1\}) \cap A = \phi \rightarrow 1 \notin d(A)$

$\because 2 \in \{2\} \in \tau \;\&\; (\{2\} - \{2\}) \cap A = \phi \Rightarrow 2 \notin d(A)$

$\because 3 \in X \in \tau \;\&\; (X - \{3\}) \cap A = \{1,2\} \cap \{1,3\} = \{1\} \neq \phi \Rightarrow$
$3 \in d(A) \Rightarrow d(A) = \{3\}$

2) Let $X = \{a,b,c,d,e,f\}$ & $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
$A = \{b,c,d,e\}$. Find ① $\overline{A}$ ② A° ③ $b(A)$ ④ $Ext(A)$ ⑤ $d(A)$

ch.

3) Let $X = \{1,2,3,4\}$ & $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$
Let $A = \{1,2,3\}$. Find ① $\bar{A}$ ② A° ③ $b(A)$ ④ $Ext(A)$ ⑤ $d(A)$ (ch.)

4) Let $(\mathbb{R}, I)$ be the indiscrete (trivial) topological space and let $A = (0,1)$, $B = \mathbb{Q}$, $C = \{2,5,7\}$ & $D = \{4\}$
Find $\bar{X}$, X°, $b(X)$, $Ext(X)$ & $d(X)$ when $X = A, B, C$ & D

Slou[t]ion

$I = \{\phi, \mathbb{R}\}$ & $\mathfrak{F} = \{\mathbb{R}, \phi\}$

① $A = (0,1)$

$\therefore \bar{A} = \cap\{F \mid F \underset{\text{closed}}{\subseteq} \mathbb{R} \ \& \ A \subseteq F\} = \mathbb{R}$

rational numbers

$B = \mathbb{Q} \Rightarrow \bar{B} = \cap\{F \mid F \underset{\text{closed}}{\subseteq} \mathbb{R} \ \& \ B \subseteq F\} = \mathbb{R}$

$\bar{C} = \mathbb{R}$ & $\bar{D} = \mathbb{R}$

② $A^\circ = \cup\{U \mid U \underset{\text{open}}{\subseteq} \mathbb{R} \ \& \ U \subseteq A\} = \phi$

$B^\circ = \phi$, $C^\circ = \phi$ & $D^\circ = \phi$

3) $b(A) = \{x \in \mathbb{R} \mid \forall U \in I \ \& \ x \in U ;\ U \cap A \neq \phi \ \& \ U \cap A^c \neq \phi\}$
Let $x \in \mathbb{R}$
$\therefore \mathbb{R}$ is the only open subset of $\mathbb{R}$ which contains x and
$\mathbb{R} \cap A = A \neq \phi$ & $\mathbb{R} \cap A^c = A^c \neq \phi \Rightarrow x \in b(A)$

$\Rightarrow b(A) = \mathbb{R}$
$b(B) = b(C) = b(D) = \mathbb{R}$ (ch.)

④ $Ext(A) = \bar{A}^c = \mathbb{R}^c = \phi$
$Ext(B) = Ext(C) = Ext(D) = \phi$ (ch.)

⑤ $d(A) = \{x \in \mathbb{R} \mid \forall U \in I \ \& \ x \in U ,\ (U - \{x\}) \cap A \neq \phi\}$
Let $x \in \mathbb{R}$

$\because \mathbb{R}$ is the only open subset of $\mathbb{R}$ which contains x and
$(\mathbb{R}-\{x\}) \cap A = (\mathbb{R}-\{x\}) \cap (0,1) \neq \phi \Rightarrow x \in d(A)$
$\therefore d(A) = \mathbb{R}$

$d(B) = d(C) = \mathbb{R}$ (ch.)

$D = \{4\} \xrightarrow{\text{To find}} d(D)$?
Let $x \in \mathbb{R}$ & $x \neq 4$
$\because \mathbb{R}$ is the only open subset of $\mathbb{R}$ which contains x and
$(\mathbb{R}-\{x\}) \cap D = (\mathbb{R}-\{x\}) \cap \{4\} \neq \phi \Rightarrow x \in d(D)$

$\therefore d(D) = \mathbb{R} - \{4\} \longrightarrow ①$

& If $x = 4 \Rightarrow 4 \notin d(D)$, since

$(\mathbb{R}-\{4\}) \cap D = (\mathbb{R}-\{4\}) \cap \{4\} = \phi \longrightarrow ②$

$\therefore$ From ① & ② we get $d(D) = \mathbb{R} - \{4\}$

⑤ Let $(\mathbb{R}, D)$ be the discrete topological space and let $A = (0,1]$
$B = \mathbb{Q}$ & $C = \{1, 4, 9\}$. Find $\overline{X}, X^{\circ}, b(X), Ext(X)$ &
$d(X)$ when $X = A, B$ & C

Solution

$D = \{U / U \subseteq \mathbb{R}\}$ & $\mathcal{F} = \{F / F \subseteq \mathbb{R}\}$
i.e any subset of $\mathbb{R}$ is open and closed
$A = (0,1]$

A closed $\Rightarrow \overline{A} = A$

① $\overline{A} = \cap \{F / F \underset{\text{closed}}{\subseteq} \mathbb{R}$ & $A \subseteq F\} = A = (0,1]$

$\overline{B} = B = \mathbb{Q}$ & $\overline{C} = C = \{1, 4, 9\}$

A open $\Rightarrow A^{\circ} = A$

② $A^{\circ} = \cup \{U / U \underset{\text{open}}{\subseteq} \mathbb{R}$ & $U \subseteq A\} = (0,1]$

$B^{\circ} = B = \mathbb{Q}$ & $C^{\circ} = C = \{1, 4, 9\}$