$\because F_1 \cup F_2 = (X - U_1) \cup (X - U_2) = X - (U_1 \cap U_2)$

$\because \tau$ is a topology $\Rightarrow U_1 \cap U_2 \in \tau \Rightarrow F_1 \cup F_2 \in \mathcal{F}$

③ Let $F_\alpha \in \mathcal{F} \ \forall \alpha \in \Lambda \Rightarrow \exists\, U_\alpha \in \tau$ s.t $F_\alpha = X - U_\alpha \ \forall \alpha \in \Lambda$

$\because \bigcap_{\alpha \in \Lambda} F_\alpha = \bigcap_{\alpha \in \Lambda} (X - U_\alpha) = X - \bigcup_{\alpha \in \Lambda} U_\alpha$

$\because \tau$ is a topology $\Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau \Rightarrow \bigcap_{\alpha \in \Lambda} F_\alpha \in \mathcal{F}$

Conversely: define τ as follows:

$\tau = \{ U \subseteq X \,/\, U = X - F,\ F \in \mathcal{F} \}$

To prove that τ is a topology on X

① $\because \phi = X - X$ & $X \in \mathcal{F} \Rightarrow \phi \in \tau$

$\because X = X - \phi$ & $\phi \in \mathcal{F} \Rightarrow X \in \tau$

$\Rightarrow \phi, X \in \tau$

② Let $U_1, U_2 \in \tau$ T.p $U_1 \cap U_2 \in \tau$

$\because U_1, U_2 \in \tau \Rightarrow U_1 = X - F_1$ & $U_2 = X - F_2$ where $F_1, F_2 \in \mathcal{F}$

$\because U_1 \cap U_2 = (X - F_1) \cap (X - F_2) = X - (F_1 \cup F_2)$

By ② $F_1 \cup F_2 \in \mathcal{F} \Rightarrow U_1 \cap U_2 \in \tau$

③ Let $U_\alpha \in \tau \ \forall \alpha \in \Lambda$ T.p $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$

$\because U_\alpha \in \tau \ \forall \alpha \in \Lambda \Rightarrow U_\alpha = X - F_\alpha$ & $F_\alpha \in \mathcal{F} \ \forall \alpha \in \Lambda$

$\because \bigcup_{\alpha \in \Lambda} U_\alpha = \bigcup_{\alpha \in \Lambda} (X - F_\alpha) = X - (\bigcap_{\alpha \in \Lambda} F_\alpha)$

By ③ $\bigcap_{\alpha \in \Lambda} F_\alpha \in \mathcal{F} \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$

$\Rightarrow \tau$ is a topology on X

Let $\mathcal{F}'$ be the collection of all τ-closed subsets of X

To prove that $\mathcal{F}' = \mathcal{F}$

Let $F' \in \mathcal{F}' \rightarrow F' = X - U \;, U \in \tau$

$\because U \in \tau \rightarrow U = X - F \;, F \in \mathcal{F}$

$\therefore F' = X - U = X - (X - F) = F \in \mathcal{F}$

$\rightarrow \mathcal{F}' \underset{\text{subColl}}{\subseteq} \mathcal{F} \longrightarrow (1)$

Also, if $F \in \mathcal{F} \rightarrow X - F \in \tau$

But $F = X - \underbrace{(X - F)}_{\in \tau} \in \mathcal{F}' \rightarrow \mathcal{F} \underset{\text{subColl}}{\subseteq} \mathcal{F}' \longrightarrow (2)$

From (1) & (2) $\rightarrow \mathcal{F}' = \mathcal{F}$

Example: Let X be any infinite set & let

$\mathcal{F} = \{F \subseteq X / F \text{ is finite}\} \cup \{X\}$

Since $\mathcal{F}$ satisfies (1), (2) & (3) of theorem (2) (ch)

then $\mathcal{F}$ induced a topology on X called the cofinite topology on X

Example: Let $X = \{a, b, c\}$ & $\tau = D$ = discrete topology

i.e $D = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$

$\mathcal{F} = \{\phi, X, \{b,c\}, \{a,c\}, \{a,b\}, \{c\}, \{b\}, \{a\}\}$

$\Rightarrow$ Any subset of discrete topology is open & closed

EX: Let $(\mathbb{R}, \mathcal{U})$ be the usual topological space

$\therefore$ open sets in $\mathbb{R}$ are $\{\phi, \mathbb{R}, (a,b), (b,\infty) \;\&\; (-\infty, a)\}$

where $a, b \in \mathbb{R}$

closed sets in $\mathbb{R}$ are $\mathcal{F} = \{\phi, \mathbb{R}, [a,b], [b,\infty), (-\infty, a]\}$

$\mathbb{Z}, \mathbb{N}$, any finite set }

$\mathbb{Z} \underset{closed}{\subseteq} \mathbb{R}$ since $\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n-1, n) \underset{open}{\subseteq} \mathbb{R}$

$\{a_1, a_2\} \underset{closed}{\subseteq} \mathbb{R}$ since $\{a_1, a_2\}^c = \mathbb{R} - \{a_1, a_2\} \underset{open}{\subseteq} \mathbb{R}$

where $\mathbb{R} - \{a_1, a_2\} = \underbrace{(-\infty, a_1) \cup (a_1, a_2) \cup (a_2, \infty)}_{open}$ $\quad a_1 \quad a_2 \quad \mathbb{R}$

Ex Let $(\mathbb{R}, \tau_{cof.})$ be the cofinite topological space

$\tau_{cof.} = \{U / U \subseteq \mathbb{R} \ \& \ U^c \text{ is finite}\} \cup \{\phi\}$

If $U \in \tau_{cof.}$ ($U \underset{open}{\subseteq} \mathbb{R}$) $\Rightarrow U^c$ is finite $\Rightarrow$ any finite set in $\mathbb{R}$ is closed

$[0,1]$ is not open in $\mathbb{R}$, since $[0,1]^c$ is not finite

$(-2, 10)$ is not open in $\mathbb{R}$, since $(-2, 10)^c$ is not finite

$\mathbb{R} - \{0\}$ is open in $\mathbb{R}$, since $(\mathbb{R} - \{0\})^c = \{0\}$ is finite

Remarks: ① The finite union of τ-closed (closed) subsets of X is τ-closed (closed). (h)

② The union of any family of τ-closed subsets of X may not be τ-closed.

Example: Let $(\mathbb{R}, \mathcal{U})$ be the usual topological space

$\forall x \in (0,1) \Rightarrow \{x\} \underset{closed}{\subseteq} \mathbb{R}$ (since $\{x\}$ is a finite set)

$\therefore \bigcup_{x \in (0,1)} \{x\} = (0,1)$, but $(0,1)$ is not closed in $\mathbb{R}$

Def: Let X be any non-empty set and let τ_1 and τ_2 be two topologies on X, we say that τ_1 is finer (larger or stronger) than τ_2 ($\tau_2 \subseteq \tau_1$) iff each τ_2-open subset of X is a τ_1-open subset of X. τ_2 is said to be coarser (weaker or smaller) than τ_1.

Examples: ① It is clear that the discrete topology is finer than any other topology on a set X and the indiscrete (trivial) topology is coarser than any other topology

② Let $X=\{1,2,3\}$

$$\left.\begin{array}{l} \tau_1 = I = \{\phi, X\} \\ \tau_2 = D = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\} \\ \tau_3 = \{\phi, X, \{1\}\} \\ \tau_4 = \{\phi, X, \{1\}, \{2\}, \{1,2\}\} \end{array}\right\} \text{are topologies on } X$$

$\Rightarrow$ τ_2 is finer than any other topologies on X

i.e $\tau_1 \subseteq \tau_2$, $\tau_3 \subseteq \tau_2$ & $\tau_4 \subseteq \tau_2$

τ_1 is coarser than any other topology on X

i.e $\tau_1 \subseteq \tau_2$, $\tau_1 \subseteq \tau_3$ & $\tau_1 \subseteq \tau_4$

τ_4 is finer than τ_3, since $\tau_3 \subseteq \tau_4$

& τ_3 is coarser than τ_4, since $\tau_3 \subseteq \tau_4$

Remark: $\tau_1 = \tau_2$ iff τ_1 is finer than τ_2 and τ_2 is finer than τ_1