i.e $d: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ defined by $d(x,y)=|x-y| \ \forall x,y \in \mathbb{R}$

$\mathcal{U}=\tau_d=\{U\subseteq \mathbb{R} \,/\, \forall x\in U, \exists \epsilon>0 \text{ s.t } N_\epsilon(x) \subseteq U\}$

$N_\epsilon(x)=\{y\in\mathbb{R} \,/\, d(x,y)<\epsilon\}=\{y\in\mathbb{R} \,/\, |y-x|<\epsilon\}$
$=\{y\in\mathbb{R} \,/\, -\epsilon<y-x<\epsilon\}=\{y\in\mathbb{R} \,/\, x-\epsilon<y<x+\epsilon\}$
$=(x-\epsilon, x+\epsilon)$

$\longrightarrow N_\epsilon(x)=(x-\epsilon, x+\epsilon)$

$\therefore \mathcal{U}=\{U\subseteq \mathbb{R} \,/\, \forall x\in U, \exists \epsilon>0 \text{ s.t } (x-\epsilon, x+\epsilon)\subseteq U\}$

$\longrightarrow$ $\mathcal{U}$ is a topology on $\mathbb{R}$ called the usual topology on $\mathbb{R}$ & is denoted by $(\mathbb{R},\mathcal{U})$

8) In example 6 take $X=\mathbb{R}^n$ and d to be the pythagorean metric
i.e $d: \mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$
$\forall (x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$
$\mathcal{U}=\tau_d=\{U\subseteq\mathbb{R}^n \,/\, \forall x\in U, \exists\epsilon>0 \text{ s.t } N_\epsilon(x)\subseteq U\}$

$\longrightarrow$ $\mathcal{U}$ is a topology on $\mathbb{R}^n$ called the usual topology on $\mathbb{R}^n$

9) Let $\mathbb{N}$ be the set of all natural numbers and let
$\tau=\{\phi\}\cup\{A_n \,/\, n\in\mathbb{N}\}$ where $A_n=\{n,n+1,n+2,\dots\}$
Then τ is a topology on $\mathbb{N}$ (ch.)

Solution

1) $\phi\in\tau$ by definition of τ
$\therefore A_1=\{1,2,3,\dots\}=\mathbb{N}\in\tau$ $\Big\} \longrightarrow \phi,\mathbb{N}\in\tau$

2) Let $U_1,U_2\in\tau$ T.p $U_1\cap U_2\in\tau$
$\therefore U_1,U_2\in\tau \Longrightarrow U_1=A_i$ and $U_2=A_j$ where $i,j\in\mathbb{N}$

$$U_1\cap U_2=\begin{cases} A_i & i\geq j \\ A_j & j\geq i \end{cases}$$

If $U_1 = \phi$ or $U_2 = \phi \Rightarrow U_1 \cap U_2 = \phi \Rightarrow U_1 \cap U_2 \in \tau$

③ If $U_\alpha \in \tau \ \forall \alpha \in \Lambda \Rightarrow U_\alpha = A_{n_\alpha}$ for some $n_\alpha \in \mathbb{N}$ or $U_\alpha = \phi$

$\therefore \bigcup_{\alpha \in \Lambda} U_\alpha = A_i$ where $i = \min\{n_\alpha\}_{\alpha \in \Lambda} \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$

$\Rightarrow \tau$ is a topology on $\mathbb{N}$

10) Let $\mathbb{R}$ be the set of all real numbers and $\forall a \in \mathbb{R}$, let $E_a = (a, \infty)$ & $\tau_{ray} = \{E_a / a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$

Then τ_{ray} is a topology on $\mathbb{R}$ called the ray topology and is denoted by $(\mathbb{R}, \tau_{ray})$ ch

Th ① The intersection of any family of topologies on a set X is a topology on X ch

proof

Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on X

T.P $\bigcap_{\alpha \in \Lambda} \tau_\alpha$ is a topology on X

① $\because \tau_\alpha$ is a topology on $X \ \forall \alpha \in \Lambda \Rightarrow \phi, X \in \tau_\alpha \ \forall \alpha \in \Lambda$

$\Rightarrow \phi, X \in \bigcap_{\alpha \in \Lambda} \tau_\alpha$

② Let $U_1, U_2 \in \bigcap_{\alpha \in \Lambda} \tau_\alpha$ T.P $U_1 \cap U_2 \in \bigcap_{\alpha \in \Lambda} \tau_\alpha$

$\because U_1, U_2 \in \bigcap_{\alpha \in \Lambda} \tau_\alpha \Rightarrow U_1, U_2 \in \tau_\alpha \ \forall \alpha \in \Lambda$

$\because \tau_\alpha$ is a topology on $X \ \forall \alpha \in \Lambda \Rightarrow U_1 \cap U_2 \in \tau_\alpha \ \forall \alpha \in \Lambda$

$\Rightarrow U_1 \cap U_2 \in \bigcap_{\alpha \in \Lambda} \tau_\alpha$

3) Let $U_\beta \in \bigcap_{\alpha\in\Lambda} \tau_\alpha \;\; \forall \beta \in \Omega$ T.p $\bigcup_{\beta\in\Omega} U_\beta \in \bigcap_{\alpha\in\Lambda} \tau_\alpha$

$\because U_\beta \in \bigcap_{\alpha\in\Lambda} \tau_\alpha \;\; \forall \beta\in\Omega \rightarrow U_\beta \in \tau_\alpha \;\; \forall \alpha\in\Lambda \;\&\; \forall \beta\in\Omega$

$\because \tau_\alpha$ is a topology on X $\forall \alpha\in\Lambda \rightarrow \bigcup_{\beta\in\Omega} U_\beta \in \tau_\alpha \;\; \forall \alpha\in\Lambda$

$\Rightarrow \bigcup_{\beta\in\Omega} U_\beta \in \bigcap_{\alpha\in\Lambda} \tau_\alpha$

$\Rightarrow \bigcap_{\alpha\in\Lambda} \tau_\alpha$ is a topology on X

Remarks:

① If τ_1 and τ_2 are two topologies on a non-empty set X, then $\tau_1 \cup \tau_2$ may not be a topology on X

Example: Let $X = \{a, b, c\}$

$\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$ are topologies on X

But $\tau_1 \cup \tau_2 = \{\phi, X, \{a\}, \{b\}\}$ is not a topology on X, since $\{a\}, \{b\} \in \tau_1 \cup \tau_2$, but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$

② The intersection of any family of τ-open subsets of X may not be τ-open

Example: Let $(\mathbb{R}, \mathcal{U})$ be the usual topological space

and let $\{(-\frac{1}{n}, \frac{1}{n})\}_{n\in\mathbb{N}}$ be the family of τ-open subsets of $\mathbb{R}$

$\Rightarrow (-\frac{1}{n}, \frac{1}{n}) \underset{open}{\subseteq} \mathbb{R} \;\; \forall n\in\mathbb{N}$

$\because \bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$, but $\{0\}$ is not open in $\mathbb{R}$

since $0 \in \{0\}$ & $\forall \epsilon > 0 \rightarrow 0 \in (-\epsilon, \epsilon) \not\subseteq \{0\}$

③ The finite intersection of τ-open subsets of X is τ-open

Definition: Let (X,τ) be a topological space. A subset F of X is called τ-closed (closed) subset of X if $F = X - U$ for some τ-open subset U of X. The collection of all τ-closed subsets of X is denoted by $\mathcal{F}$

Examples ① In $(X,I) \Rightarrow \mathcal{F} = \{\phi, X\}$

② In $(X,D) \Rightarrow \mathcal{F} = \{F \mid F \subseteq X\}$

③ $X = \{1,2,3\}$ & $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

$\Rightarrow \mathcal{F} = \{\phi, X, \{2,3\}, \{1,3\}, \{3\}\}$

Theorem (2) Let (X,τ) be a topological space and $\mathcal{F}$ be the collection of all τ-closed subsets of X, then

① $\phi, X \in \mathcal{F}$

② If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$

③ If $F_\alpha \in \mathcal{F}$ $\forall \alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} F_\alpha \in \mathcal{F}$

Conversely: If $X \neq \phi$ and $\mathcal{F}$ is a collection of subsets of X satisfying ①, ② and ③ then $\mathcal{F}$ induced a topology τ on X such that the collection of all τ-closed subsets of X is $\mathcal{F}$

Proof: $\Rightarrow$

① $\because X = X - \phi$ & $\phi \in \tau \Rightarrow X \in \mathcal{F}$

$\because \phi = X - X$ & $X \in \tau \Rightarrow \phi \in \mathcal{F}$

$\therefore \phi, X \in \mathcal{F}$

② Let $F_1, F_2 \in \mathcal{F}$ T.P $F_1 \cup F_2 \in \mathcal{F}$

$\because F_1, F_2 \in \mathcal{F} \Rightarrow \exists\, U_1, U_2 \in \tau$ s.t $F_1 = X - U_1$ & $F_2 = X - U_2$