Topological spaces

Definition: Let X be any non-empty set. A collection τ of subsets of X is called a topology on X if it satisfies the following axioms:

① ϕ and X are elements of τ.

② The intersection of any two (finite) elements of τ is an element of τ
i.e. if $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$

③ The union of any family of elements of τ is an element of τ
i.e. if $U_\alpha \in \tau \ \forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$

The elements of τ are called τ-open (open) subsets of X. The set X with τ is called a topological space and is denoted by (X, τ).

Examples: ① Let $X = \{1,2,3\}$ and let $\tau_1 = \{\phi, X, \{1\}, \{1,2\}\}$
$\tau_2 = \{\phi, X, \{1\}, \{3\}\}$, $\tau_3 = \{\phi, X, \{1,2\}, \{2,3\}\}$
$\tau_4 = \{\phi, \{2\}, \{2,3\}\}$ and $\tau_5 = \{X, \{2\}, \{3\}, \{2,3\}\}$

It is clear that τ_1 is a topology on X and (X, τ_1) is a topological space
τ_2 is not a topology on X since $\{1\}, \{3\} \in \tau_2$, but $\{1\} \cup \{3\} = \{1,3\} \notin \tau_2$
τ_3 " " " " " " $\{1,2\}, \{2,3\} \in \tau_3$, but $\{1,2\} \cap \{2,3\} = \{2\} \notin \tau_3$
τ_4 " " " " " since $X \notin \tau_4$
τ_5 " " " " " since $\phi \notin \tau_5$

② Let $X \neq \phi$ and $\tau = \{\phi, X\} = I$, then τ is a topology on X called the indiscrete topology (or trivial topology)

③ Let $X \neq \phi$ and $\tau = \{U \mid U \subseteq X\} = P(X) = D$, then τ is
power set of X

a topology on X called the discrete topology on X.

Solutions

1) $\because \phi \subseteq X \quad \forall X \Rightarrow \phi \in D$
$\because X \subseteq X \quad \forall X \Rightarrow X \in D$ } $\Rightarrow$ ϕ and X are elements of τ

2) Let $U_1, U_2 \in \tau$ T.p $U_1 \cap U_2 \in \tau$
$\because U_1 \in \tau \Rightarrow U_1 \subseteq X$
$\because U_2 \in \tau \Rightarrow U_2 \subseteq X$
$\Rightarrow U_1 \cap U_2 \subseteq X \Rightarrow U_1 \cap U_2 \in \tau$

3) Let $U_\alpha \in \tau \ \forall \alpha \in \Lambda$ T.p $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$

$\because U_\alpha \in \tau \ \forall \alpha \in \Lambda \Rightarrow U_\alpha \subseteq X \ \forall \alpha \in \Lambda \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \subseteq X$

$\Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$

$\Rightarrow \tau = D$ is a topology on X and (X, D) is called the discrete topological space.

4) Let X be any infinite set and let
$\tau = \{U \mid U \subseteq X, U^c \text{ is finite}\} \cup \{\phi\}$
Then τ is a topology on X called the cofinite topology and is denoted by τ_{cof}.

Solutions

1) $\phi \in \tau_{cof}$ by definition of τ_{cof}.

$\because X \subseteq X$ and $X^c = \phi$ is finite $\Rightarrow X \in \tau_{cof}$.

2) Let $U_1, U_2 \in \tau_{cof}$ T.p $U_1 \cap U_2 \in \tau_{cof}$.

If $U_1 \cap U_2 = \phi \Rightarrow \phi \in \tau_{cof}$.

If $U_1 \cap U_2 \neq \phi \Rightarrow U_1 \neq \phi$ and $U_2 \neq \phi$
$\because U_1 \in \tau_{cof} \Rightarrow U_1 \subseteq X$ and U_1^c is finite

~~$\because U_2 \in \tau_{cof.} \Rightarrow U_2 \subseteq X$ and U_2^c is finite~~

$\because U_1 \cap U_2 \subseteq X$ and $(U_1 \cap U_2)^c = \underset{\text{finite}}{U_1^c} \cup \underset{\text{finite}}{U_2^c}$ is finite $\Rightarrow U_1 \cap U_2 \in \tau_{cof.}$

③ Let $U_\alpha \in \tau_{cof.}$ $\forall \alpha \in \Lambda$ T.P $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_{cof.}$

$\because U_\alpha \in \tau_{cof.}$ $\forall \alpha \in \Lambda \Rightarrow U_\alpha \subseteq X$ and U_α^c is finite $\forall \alpha \in \Lambda$

$\therefore \bigcup_{\alpha \in \Lambda} U_\alpha \subseteq X$ and $\left(\bigcup_{\alpha \in \Lambda} U_\alpha\right)^c = \bigcap_{\alpha \in \Lambda} U_\alpha^c$ is finite $\Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_{cof.}$

$\Rightarrow \tau_{cof.}$ is a topology on X

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⑤ Let X be any uncountable set and let
$\tau = \{U \mid U \subseteq X, U^c \text{ is countable}\} \cup \{\phi\}$
Then τ is a topology on X called the cocountable topology and is denoted by $\tau_{coc.}$

⑥ Let (X, d) be any metric space and let
$\tau_d = \{U \mid U \subseteq X, \forall x \in U, \exists \epsilon > 0 \text{ s.t } N_\epsilon(x) \subseteq U\}$

where $N_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$

$N_\epsilon(x)$ is called an open ball of radius ϵ and center x.

Then τ_d is a topology on X called the metric topology on X

Solutions

$\phi \subseteq X$

① Since $\nexists x \in \phi \Rightarrow \forall \epsilon > 0, N_\epsilon(x) \subseteq \phi \Rightarrow \phi \in \tau_d$

$X \subseteq X$, since $\forall x \in X$ & $\forall \epsilon > 0$, $N_\epsilon(x) \subseteq X$
$\Rightarrow X \in \tau_d$
$\Rightarrow \phi, X \in \tau_d$

② Let $U_1, U_2 \in \tau_d$ T.p $U_1 \cap U_2 \in \tau_d$

if $U_1 \cap U_2 = \phi \longrightarrow U_1 \cap U_2 \in \tau_d$

if $U_1 \cap U_2 \neq \phi \longrightarrow \exists\, x \in U_1 \cap U_2 \longrightarrow x \in U_1$ and $x \in U_2$

$\because U_1, U_2 \in \tau_d \longrightarrow \exists\, \epsilon_1, \epsilon_2 > 0$ s.t $N_{\epsilon_1}(x) \subseteq U_1$ & $N_{\epsilon_2}(x) \subseteq U_2$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ T.p $N_\epsilon(x) \subseteq U_1 \cap U_2$

Let $y \in N_\epsilon(x)$ T.p $y \in U_1 \cap U_2$

$\because y \in N_\epsilon(x) \longrightarrow d(x,y) < \epsilon$

$\because \epsilon = \min\{\epsilon_1, \epsilon_2\} \longrightarrow d(x,y) < \epsilon_1$ and $d(x,y) < \epsilon_2$

$\longrightarrow y \in N_{\epsilon_1}(x)$ and $y \in N_{\epsilon_2}(x)$

$\because N_{\epsilon_1}(x) \subseteq U_1$ and $N_{\epsilon_2}(x) \subseteq U_2 \longrightarrow y \in U_1$ and $y \in U_2$

$\longrightarrow y \in U_1 \cap U_2$

$\longrightarrow N_\epsilon(x) \subseteq U_1 \cap U_2$

$\Longrightarrow U_1 \cap U_2 \in \tau_d$

③ Let $U_\alpha \in \tau_d \;\forall \alpha \in \Lambda$ T.p $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_d$

Let $x \in \bigcup_{\alpha \in \Lambda} U_\alpha \longrightarrow x \in U_{\alpha_0}$ for some $\alpha_0 \in \Lambda$

$\because U_{\alpha_0} \in \tau_d \longrightarrow \exists\, \epsilon > 0$ s.t $N_\epsilon(x) \subseteq U_{\alpha_0}$

$\because U_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \longrightarrow N_\epsilon(x) \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \longrightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_d$

$\longrightarrow \tau_d$ is a topology on X.

⑦ In example ⑥ take $X = \mathbb{R}$ and d to be the absolute value metric