Lecture 2

Some Introductory Principles

Introduction

Atmospheric Dynamics is the study of those motions of the atmosphere that are associated with weather and climate.

For all such motions, the discrete molecular nature of the atmosphere can be ignored, and the atmosphere can be regarded as a continuous fluid medium, or continuum. A "point" in the continuum is regarded as a volume element that is very small compared with the volume of atmosphere under consideration, but still contains a large number of molecules.

The expressions air parcel and air particle are both commonly used to refer to such a point. The various physical quantities that characterize the state of the atmosphere (e.g., pressure, density, temperature) are assumed to have unique values at each point in the atmospheric continuum. Moreover, these field variables and their derivatives are assumed to be continuous functions of space and time.

The fundamental laws of fluid mechanics and thermodynamics, which govern the motions of the atmosphere, may then be expressed in terms of partial differential equations involving the field variables as dependent variables and space and time as independent variables.

The general set of partial differential equations governing the motions of the atmosphere is extremely complex; no general solutions are known to exist. To acquire an understanding of the physical role of atmospheric motions in determining the observed weather and climate, it is necessary to develop models based on systematic simplification of the fundamental governing equations.

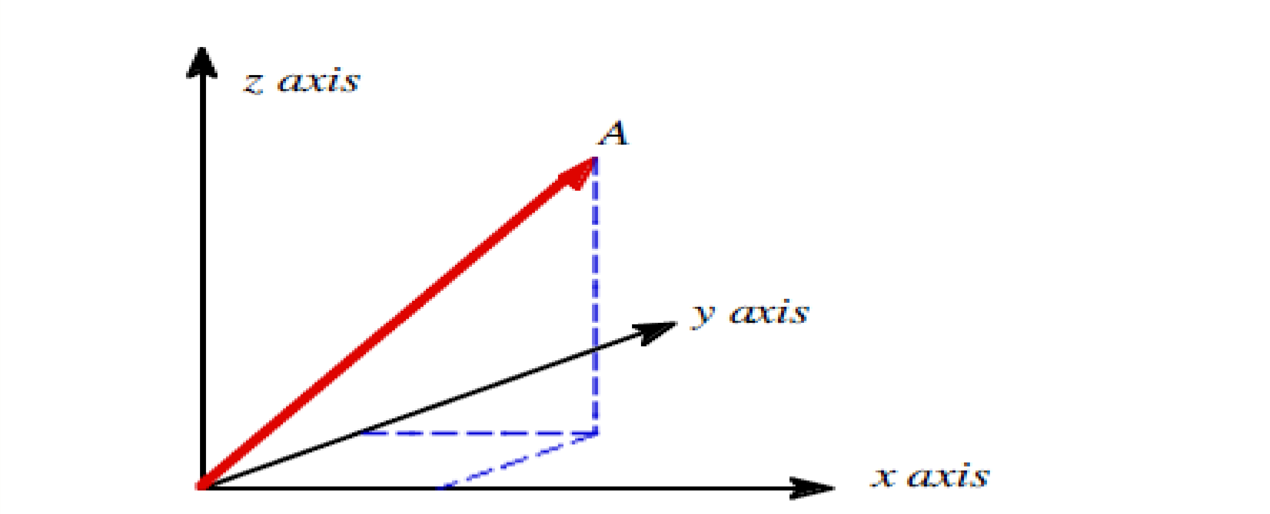
This lecture will give an introduction to vector analysis. A vector, or displacement, has a direction and a magnitude. The following vector notations will be used in this course: 1. A : the vector A 2. A : the magnitude of vector A The opposite of a vector A is a vector with the same magnitude as A but pointing in a direction opposite to that of A . The opposite of vector A is written as -A .

1- **Cartesian coordinate**

Vector components A vector in three dimensions can be identified uniquely by specifying three coordinates. **In a Cartesian coordinate** frame three base vectors ˆ i , ˆ j , and ˆ k are defined, parallel to the x, y, and z axes, respectively (see Figure 1). Each of these base vectors has unit length and is perpendicular to the other two base vectors. Any vector is uniquely defined by specifying its components along the x, y, and z axes.

A vector A is defined in terms of the three Cartesian coordinates Ax, Ay, and Az . Using the three base vectors, one can reconstruct the vector A :

$$\bar{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$



Curvilinear coordinates

The Cartesian coordinate system is a coordinate system that is often used in calculations involving systems with no apparent symmetry. To describe systems that have spherical or cylindrical symmetry it is often more convenient to use spherical coordinates or cylindrical coordinates, respectively. These two coordinate systems will be discussed in this section.

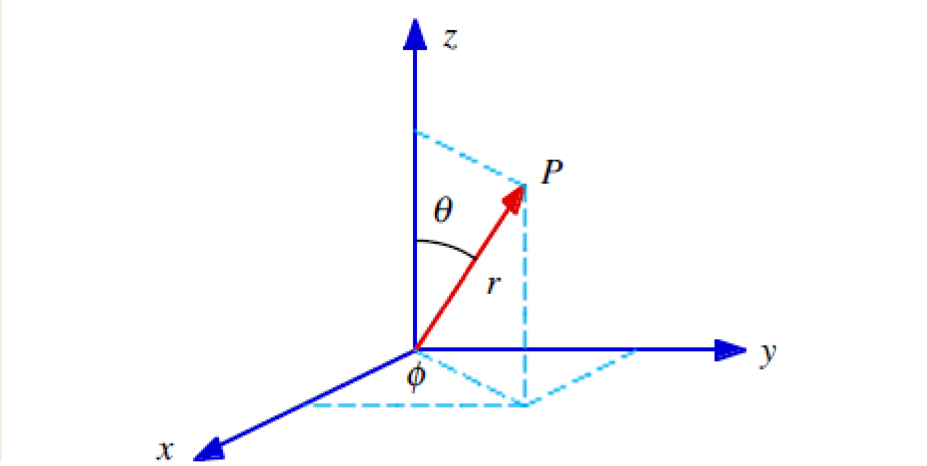
2- <u>**Spherical coordinates**</u>

Spherical coordinates are always used when the system under consideration has spherical symmetry. The location of a point P (see Figure 2) is completely determined by specifying the following three coordinates:

1. r : the distance between the origin and P.

2. q : the polar angle which is the angle between the vector P and the z-axis (see Figure 2).

3. f : the azimuthal angle which is the angle between the projection of the vector to P in the xy



In general, any vector $\overline{A}$ can be expressed in terms of these three coordinates:

$$\overline{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

In contrast to the unit vectors i ˆ , ˆj , and k ˆ in a Cartesian coordinate system, the unit vectors r ˆ ,q ˆ , and f ˆ in a spherical coordinate system are not constant; they change direction as P moves around.

Consider a point P (see Figure 2) which is defined by the three spherical coordinates (r,q , and f). The corresponding Cartesian coordinates are

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

3- Conversion Spherical to Cartesian Coordinates:

Conversions between increments of distance in Cartesian coordinates and increments of longitude or latitude in spherical coordinates, along the surface of Earth, are obtained from the equation for arc length around a circle. In the west–east and south–north directions, these conversions are:

$$dx = (R_e \cos\phi)d\lambda_e \qquad dy = R_e\, d\phi \qquad (1.1)$$

where $R_e = 6371$ km is the radius of the Earth, $d\lambda_e$ is a west-east longitude increment (radians), $d\phi$ is a south-north latitude increment (radians).

Example 1.1: If a grid cell has dimensions $d\lambda_e = 5^o$ and $d\phi = 5^o$, centered at $\phi = 30^oN$ latitude, find dx and dy at the grid cell latitudinal center.

Solution: $dx = (R_e \cos\phi)d\lambda_e \qquad dy = R_e\, d\phi$

First, $d\lambda_e = d\phi = 5^o \times \pi/180^o = 0.0873$ radians.

Hence: $dx = (6371)(0.866)(0.0873) = 482km$ and

$dy = (6371)(0.0873) = 556km.$

4- **Wind Velocity**: Winds are described by three parameters: *velocity*, *the scalar components of velocity*, and *speed*. Velocity is a vector that quantifies the rate at which the position of a body changes over time:

$$\vec{V} = iu + jv + kw \quad (total\ vector) \qquad (1.2)$$

$$\vec{V}_h = iu + jv \quad (horizontal\ vector) \qquad (1.3)$$

And, $$u = \frac{dx}{dt}\ ,\ v = \frac{dy}{dt}\ ,\ w = \frac{dz}{dt} \qquad (1.4)$$

are the scalar components of velocity. They have magnitude only. The magnitude of the wind is its speed. The total and horizontal wind speeds are defined as:

$$|\vec{V}| = \sqrt{u^2 + v^2 + w^2} \quad , \quad |\vec{V}_h| = \sqrt{u^2 + v^2} \qquad (1.5)$$

5- Vector Multiplication:

There are two types of multiplication, namely:

- *The dot product*: It is a product of two vectors gives a scalar.

Let $\vec{A}$ and $\vec{B}$ are two vectors,

$$\vec{A} = iA_x + jA_y + kA_z \quad , \quad \vec{B} = iB_x + jB_y + kB_z$$

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z \quad \text{(How?)} \qquad (1.6)$$

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|cos\theta \qquad (1.7)$$

Example 1.2: Let $\vec{A} = 2i - 1/2j - 3k$, and $\vec{B} = -3i + j - 1/2k$

Find $\vec{A}.\vec{B}$ and the angle between the two vectors.

Solution:

$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$

$\vec{A}.\vec{B} = -6 + (-1/2) + 3/2 \quad = -6.5 + 1.5 = -5$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{4 + 1/4 + 9} = \sqrt{13.75}$$

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{9 + 1 + 1/4} = \sqrt{10.25}$$

$$cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{-5}{\sqrt{13.25}\sqrt{10.25}} = \frac{-5}{11.52} = -0.434$$

$$\theta = cos^{-1}\theta = 115$$

- *The Cross Product*: it is product of vectors gives a vector:

$$\vec{A} \times \vec{B} = \vec{C} \qquad (1.8)$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin\theta \qquad (1.9)$$

The direction of $\vec{C}$ is perpendicular on the plane of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \vec{C} = (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k \qquad (1.10)$$

6- Gradient, Divergence and Curl Operators

Del Operator: is a vector differential operator denoted by the symbol $\vec{\nabla}$:

$$\vec{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad (1.11)$$

This operator can be used in three differential ways:

a. Gradient of Scalar (pressure)

$$\vec{\nabla}p = i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y} + k\frac{\partial p}{\partial z} \qquad (vector) \qquad (1.12)$$

b. Divergence of a Vector (velocity)

$$\vec{\nabla}.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (scalar)\ (How?) \qquad (1.13)$$

c. Curl of a Vector (velocity)

$$\vec{\nabla}\times\vec{V}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)i+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)j+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)k \quad (vector) \quad (1.14)$$

7- Laplacian Operator

If Q is any quantity then,

$$\vec{\nabla}.\vec{\nabla}Q \equiv \vec{\nabla}^2 Q \qquad (1.15)$$

where $\vec{\nabla}^2$ (del squared) is the scalar differential operator:

$$\vec{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \qquad (1.16)$$

$\vec{\nabla}^2 Q$ is called the Laplacian of Q and appears in several important partial equations of mathematical physics.

Eulerian and Lagrangian frames of reference

Eulerian and Lagrangian frames of reference There are two ways we can describe the motion (or flow) of a fluid such as the Earth's atmosphere: the Eulerian and the Lagrangian. In an Eulerian frame of reference, the flow quantities such as temperature or velocity are defined as functions of position in space and time (for example, Figure 2.9). The primary flow quantity is the velocity vector field, but the complete description includes the spatial distribution of other quantities of interest such as temperature, pressure, and density. A flow variable is written as a function of position and time, F(x, y, z, t), and the partial derivative gives only the local rate of change at a particular location and time. So, for example, in Figure 2.9, the temperature of the smoke flowing from a chimney can be expressed using an Eulerian frame of reference as a function T(x, y, z, t), and the temperature at location O will be $T_O = Tx_o\ y_o\ z_o t$.

Typically, in the Eulerian description, the components of the wind field are designated u =u (x y z t) i +v (x y z t) j +w (x y z t) k or simply u = u v w. This standard usage for wind components will be used in the rest of this book.

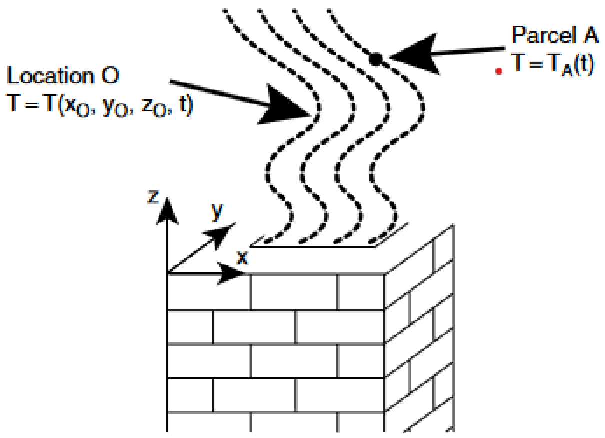


Figure 2.9 Eulerian (0) and Lagrangian (A) descriptions of the temperature of smoke from a chimney

The Lagrangian specification makes use of the fact that, as in particle mechanics, some of the dynamical and physical quantities refer not only to certain positions in space but also (and more fundamentally) to identifiable pieces of matter. The flow quantities here are defined as functions of time and the choice of the piece of matter, or parcel, and thus describe the dynamical history of the selected parcel. In this description then, any flow variable (including the location of a parcel) is expressed as a function of time only, F(t) .

Since parcels change shape as they move, parcels must be chosen such that they are considered to be ‘small’, and that ‘smallness’ must continue throughout time. So, for example, in Figure 2.9, the temperature of a parcel of smoke flowing from the chimney can be expressed using a Lagrangian specification as a function of time T(t) , and the temperature of parcel A will be T(t) .

The Lagrangian description is useful in some contexts, such as the tracking of air pollution, and may appear to be simpler. However, it can become cumbersome when there are many parcels to be tracked, such as within a large cyclone.

1.3 The Total Derivative

Meteorological variables such as $P, T, \vec{V}$ etc. can vary both in space and time, i.e. functions of four independent variables, x, y, z and t.

The differential of any of these variables (e.g., T) has the form

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz \qquad (1.17)$$

Dividing by dt,

$$\frac{dT}{dt} = \frac{\partial T}{\partial t}\frac{dt}{dt} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

By definition: $\frac{dx}{dt} \equiv u$; $\frac{dy}{dt} \equiv v$; $\frac{dz}{dt} \equiv w$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

which can also be written as,

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{V}.\nabla T \qquad (1.18)$$

This means

$$Total\ derivative = Partial\ derivative + \boldsymbol{Advection\ operator}$$

The total derivative $(\frac{dT}{dt})$ represents the change relative to a reference frame attached to the air parcel and moving with it (Lagrangian derivative). The term $(\frac{\partial T}{\partial t})$ represents the change from a coordinate system fixed x, y, and z coordinates. This is called the local derivative, or the Eulerian derivative. The term $(\vec{V}.\nabla T)$ represents that part of the local change that is due to advection (transport of a property due to the mass movement of the fluid).

Rewrite equation (1.18) in the following form:

$$\frac{\partial T}{\partial t} = \frac{dT}{dt} - \vec{V}.\nabla T \qquad (1.19)$$

Enable us to understand the change we measure with our instruments:

The local measure that we make may be due to either a change within the fluid itself ($\frac{dT}{dt}$), or due to movement of the fluid with different property over our instrument, represented by ($-\vec{V}.\nabla T$) term.

Example: The temperature at our station has been decreasing. This may be due the entire air mass losing heat due to radiation or conduction ($\frac{dT}{dt}$) or due to the wind blowing colder air into our area, ($-\vec{V}.\nabla T$)