



# Foundation of Mathematics 1

## ***CHAPTER 1 LOGIC THEORY***

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## Course Outline First Semester

**Course Title:** Foundation of Mathematics 1

**Code subject:**

**Instructors:** Mustansiriyah University-College of Science-Department of Mathematics

**Stage:** The First

## Contents

Chapter 1	Logic Theory	Logic, Truth Table, Tautology, Contradiction, Contingency, Rules of Proof, Logical Implication, Quantifiers, Logical Reasoning, Mathematical Proof.
Chapter 2	Sets	Definitions, Equality of Sets, Set Laws
Chapter 3	Relations on Set	Cartesian Product, Relations, Mappings.

## References

- [1] Max D. Larsen. Fundamental Concepts of Modern Mathematics. 1970.  
[2] اساس الرياضيات, الجزء الاول. تاليف د. هادي جابر مصطفى, رياض شاکر نعوم و نادر جورج منصور.  
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### Recommended Texts

- [1] Elliott Mendelson. Introduction to Mathematical Logic, 4<sup>th</sup> edition. 1997.  
[2] Herbert B. Enderton. A Mathematical Introduction to Logic, 2<sup>nd</sup> edition. 2001.  
[3] Calvin Jongsma. Introduction to Discrete Mathematics via Logic and Proof. Springer Nature Switzerland AG, 2019.

## THE GREEK ALPHABET

<i>letter</i>	<i>name</i>	<i>capital</i>
α	<b>Alpha</b>	A
β	<b>Beta</b>	B
γ	<b>Gamma</b>	Γ
δ	<b>Delta</b>	Δ
ε	<b>Epsilon</b>	E
ζ	<b>Zeta</b>	Z
η	<b>Eta</b>	H
θ	<b>Theta</b>	Θ
ι	<b>Iota</b>	I
κ	<b>Kappa</b>	K
λ	<b>Lambda</b>	Λ
μ	<b>Mu</b>	M
ν	<b>Nu</b>	N
ξ	<b>Xi</b>	Ξ
ο	<b>Omicron</b>	O
π	<b>Pi</b>	Π
ρ	<b>Rho</b>	P
σ ς	<b>Sigma</b>	Σ
τ	<b>Tau</b>	T
υ	<b>Upsilon</b>	Υ
φ	<b>Phi</b>	Φ
χ	<b>Chi</b>	X
ψ	<b>Psi</b>	Ψ
ω	<b>Omega</b>	Ω

# Chapter One

## Logic Theory

### 1.1. Logic

#### Definition 1.1.1.

(i) **Logic** is the theory of systematic reasoning, and symbolic logic is the formal theory of logic.

(ii) A **logical proposition (statement or formula)** is a declarative sentence that is either true (denoted either T or 1) or false (denoted either F or 0) but not both.

(iii) The truth or falsehood of a logical proposition is called its **truth value**.

**Notation:** Variables are used to represent logical propositions. The most common variables used are p, q, and r.

#### Example 1.1.2.

$x + 2 = 2x$  when  $x = -2$ .

All cars are brown.

$2 \times 2 = 5$ .

Here are some sentences that are not logical propositions (**paradox**).

Look out! (Exclamatory)

How far is it to the next town? (Interrogative)

$x + 2 = 2x$ .

“Do you want to go to the movies?” (Interrogative)

“Clean up your room.” (Imperative)



**OR ( $\vee$ ) (disjunction):** these statements are false only when both p and q are false.

OR $\vee$ (Disjunction)		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

**Exclusive( $\veebar$ ) one of p or q (read p or else q)**

$\veebar$ (Exclusive)		
p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

**If  $\rightarrow$  Then Statements** – These statements are false only when p is true and q is false (because anything can follow from a false premise).

If $\rightarrow$ Then		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Here, p called **hypothesis (antecedent)** and q called **consequent (conclusion)**.

➤ Equivalent Forms of ( $p \rightarrow q$ ) read as:

- 1- If p then q”:
- 2- p implies q
- 3- p is a sufficient condition for q
- 4- p only if q=if not q then not p.
- 5- q if p.
- 6- q whenever p
- 7- q is a necessary condition for p.  
(Existence of O is necessary to exist of H<sub>2</sub>O)
- 8- q follows from p.
- 9-q, provided that p.

To understand why the conditional statement is false only in the case when p is true but q is false considering the following example:

➤ Suppose your dad promises you:

“If you get an A in Foundation1, then I will buy you a laptop”.

Here, p is “you get an A in Foundation1”,  
q is “I will buy you a laptop”.

Then the only situation you can accuse your dad of breaking his promise is when

you get an A in Foundation1

but (and)

your dad does not buy you a laptop.

If you do not get an A in Foundation1, then whether you dad buys you a laptop or not, you can’t say that he breaks his promise.

➤ The statement  $q \rightarrow p$  is called the **converse** of the statement  $p \rightarrow q$  and the statement  $\sim p \rightarrow \sim q$  is called the **inverse**.

For instance “if Ali is from Baghdad then Ali is from Iraq” is true, but the converse “if Ali is from Iraq then Ali is from Baghdad” may be false. The inverse “if Ali is not from Baghdad then Ali is not from Iraq” may be false.

➤ **Note** that the statements  $p \rightarrow q$  and  $q \rightarrow p$  are different.

**If and only If Statements** – These statements are true only when both p and q have the same truth (logical) values.

If $\leftrightarrow$ Then		
p	q	$p \leftrightarrow q$
<b>T</b>	<b>T</b>	<b>T</b>
<b>T</b>	<b>F</b>	<b>F</b>
<b>F</b>	<b>T</b>	<b>F</b>
<b>F</b>	<b>F</b>	<b>T</b>

**NOT ~ (negation)** The “not” is simply the opposite or complement of its original value.

NOT ~ (negation)	
P	$\sim p$
<b>T</b>	<b>F</b>
<b>F</b>	<b>T</b>

➤ Note that, the negation is meaningful when used with only one logical proposition. This is not true of the other connectives.

**Examples 1.2.4.** Write the following statements symbolically, and then make a truth table for the statements.

(i) If I go to the mall or go to the stadium, then I will not go to the gym.

(ii) If the fish is cooked, then dinner is ready and I am hungry.

**Solution.**

(i) Suppose we set

$p$  = I go to the mall

$q$  = I go to the stadium

$r$  = I will go to the gym

The proposition can then be expressed as “If  $p$  or  $q$ , then not  $r$ ,” or  $(p \vee q) \rightarrow \sim r$ .



### 1.3. Tautology /Contradiction / Contingency

**Definition 1.3.1. (Tautology)**

A tautology (theorem or lemma) is a logical proposition that is always true.

**Remark 1.3.2.** One informal way to check whether or not a certain logical formula is a theorem is to construct its truth table.

**Example 1.3.3.**  $p \vee \sim p$ .

**Definition 1.3.4. (Contradiction)**

A contradiction is a logical proposition that is always false.

**Example 1.3.5.**  $p \wedge \sim p$ .

**Definition 1.3.6. (Contingency)**

A contingency is a logical proposition that is neither a tautology nor a contradiction.

**Example 1.3.7.**

(i) The logical proposition  $(p \vee q) \rightarrow \sim r$  is a contingency. See Example 1.2.4(i).

(ii) The logical proposition  $p \vee \sim (p \wedge q)$  is a tautology.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

**Exercise 1.3.8.**

(i) Build a truth table to verify that the logical proposition

$$(p \leftrightarrow q) \wedge (\sim p \wedge q)$$

is a contradiction.

(ii) **(Low of Syllogism)** Show that the logical proposition

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

is a tautology.

**Definition 1.3.9. (Logically equivalent)**

Propositions **r** and **s** are logically equivalents if the truth tables of **r** and **s** are the same and denoted by  $r \equiv s$ .

**Example 1.3.10.** Show that

$$\sim(p \rightarrow q) \equiv p \wedge \sim q.$$

**Solution.**

Show the truth values of both propositions are identical.

p	q	$\sim q$	$p \rightarrow q$	$\sim(p \rightarrow q)$	$p \wedge \sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

**Remark 1.3.11. (Relation Between Logical Equivalent and Tautology)**

$(r \equiv s) \equiv (r \leftrightarrow s)$  is a tautology.

**Solution.**

r	s		r	s	$r \leftrightarrow s$	
T	T	$r \equiv s$	T	T	T	←
T	F		T	F	F	
F	T		F	T	F	
F	F	$r \equiv s$	F	F	T	←

From the above table of the propositions  $r \equiv s$  and  $(r \leftrightarrow s)$  is a tautology) we get that they have the same truth table.

### 1.3.12. Algebra of Logical Proposition

The logical equivalences below are important equivalences that should be memorized.

- 1-Identity Laws:  $p \wedge T \equiv p.$   
 $p \vee F \equiv p.$
- 2-Domination Laws:  $p \vee T \equiv T.$   
 $p \wedge F \equiv F.$
- 3-Idempotent Laws:  $p \vee p \equiv p.$   
 $p \wedge p \equiv p.$
- 4- Double Negation Law:  $\sim(\sim p) \equiv p.$
- 5- Commutative Laws:  $p \vee q \equiv q \vee p.$   
 $p \wedge q \equiv q \wedge p.$
- 6- Associative Laws:  $(p \vee q) \vee r \equiv p \vee (q \vee r).$   
 $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r).$
- 7- Distributive Laws:  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$   
 $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$
- 8- De Morgan's Laws:  $\sim (p \wedge q) \equiv \sim p \vee \sim q.$   
 $\sim (p \vee q) \equiv \sim p \wedge \sim q.$
- 9- Absorption Laws:  $p \wedge (p \vee q) \equiv p.$   
 $p \vee (p \wedge q) \equiv p.$   
 $p \wedge (\sim p \vee q) \equiv p \wedge q.$   
 $p \vee (\sim p \wedge q) \equiv p \vee q.$
- 10-Implication Law:  $(p \rightarrow q) \equiv (\sim p \vee q).$
- 11- Contrapositive Law:  $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p).$
- 12- Tautology:  $p \vee \sim p \equiv T.$
- 13- Contradiction:  $p \wedge \sim p \equiv F.$
- 14- Equivalence:  $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q).$
- 15-  $p \underline{\vee} q \equiv (p \vee q) \wedge \sim(p \wedge q).$



## 1.4. Rules of Proof

### 1.4.1.

#### (i) Rule of Replacement.

Any term in a logical formula may be replaced by an equivalent term.

For instance, if  $q \equiv r$ , then  $(p \wedge q) \equiv (p \wedge r)$  Rep( $q:r$ ).

#### (ii) Rule of Substitution.

A sentence which is obtained by substituting logical propositions for the terms of a theorem is itself a theorem.

For instance,  $(p \rightarrow q) \vee w \equiv w \vee (p \rightarrow q)$  Sub( $p: p \rightarrow q$ ), in Commutative Law

$p \vee w \equiv w \vee p$ .

#### (iii) Rule of Inference.

1-	$\frac{p}{p \rightarrow q}$ $\therefore q$	6-	$\frac{p \rightarrow q}{q \rightarrow r}$ $\therefore p \rightarrow r$
2-	$\frac{\sim q}{p \rightarrow q}$ $\therefore \sim p$	7-	$\frac{p \vee q}{\sim p}$ $\therefore q$
3-	$\frac{p}{\therefore p \vee r}$	8-	$\frac{p \vee q}{\sim p \vee r}$ $\therefore q \vee r$
4-	$\frac{p}{q}$ $\therefore p \wedge q$	9-	$\frac{p \rightarrow q}{r \rightarrow t}$ $\therefore p \vee r \rightarrow q \vee t$
5-	$\frac{p \wedge q}{\therefore p}$	10-	$\frac{p}{q \rightarrow r}$ $\therefore p \vee q \rightarrow p \vee r$

**Example 1.4.2.**

**(i)** Given

(1)  $p \wedge q$

(2)  $p \rightarrow \sim (q \wedge r)$

(3)  $s \rightarrow r$

---

$\therefore \sim s$

**Solution:**

- |                                      |                                      |
|--------------------------------------|--------------------------------------|
| 1- $p \wedge q$                      | 1 <sup>st</sup> hypothesis (premise) |
| 2- $p$                               | Inf. (1) Properties of $\wedge$      |
| 3- $q$                               | Inf. (1) Properties of $\wedge$      |
| 4- $p \rightarrow \sim (q \wedge r)$ | 2 <sup>nd</sup> hypothesis (premise) |
| 5- $\sim (q \wedge r)$               | Inf. (2),(4)                         |
| 6- $\sim q \vee \sim r$              | De Morgan's Law on (5)               |
| 7- $\sim r$                          | Inf. (3),(6) and Domination Laws     |
| 8- $s \rightarrow r$                 | 3 <sup>rd</sup> hypothesis (premise) |
| 9- $\sim r \rightarrow \sim s$       | Contrapositive Law                   |
| 10- $\sim s$                         | Inf. (7),(9)                         |

**(ii)** Given

(1)  $\sim (p \vee q) \rightarrow r$

(2)  $\sim p$

(3)  $\sim r$

---

$\therefore q$

**Solution:**

- 1-  $\sim (p \vee q) \rightarrow r$       1<sup>st</sup> hypothesis (premise)  
2-  $\sim r$       3<sup>rd</sup> hypothesis (premise)  
3-  $\sim r \rightarrow (p \vee q)$       Contrapositive Law and Double Negation Law  
4-  $p \vee q$       Inf. (2),(3)  
5-  $\sim p$       2<sup>nd</sup> hypothesis (premise)  
6-  $q$       Inf. (4),(5)

(iii) Given

(1)  $\sim p \rightarrow (r \wedge s)$

(2)  $p \rightarrow q$

(3)  $\sim q$  \_\_\_\_\_

$\therefore r$

**Solution:**

- 1-  $p \rightarrow q$       2<sup>nd</sup> hypothesis (premise)  
2-  $\sim q \rightarrow \sim p$       Contrapositive Law on (1)  
3-  $\sim q$       3<sup>rd</sup> hypothesis (premise)  
4-  $\sim p$       Inf. (2),(3)  
5-  $\sim p \rightarrow (r \wedge s)$       1<sup>st</sup> hypothesis (premise)  
6-  $r \wedge s$       Inf. (4),(5)  
7-  $r$       Inf. (6) Properties of  $\wedge$

(iv) Given

$$(1) p \rightarrow (\sim r \wedge \sim s)$$

$$(2) p \vee \sim q$$

$$(3) s$$

---

$$\therefore \sim q \wedge s$$

**Solution:**

$$1- p \rightarrow (\sim r \wedge \sim s) \quad 1^{\text{st}} \text{ hypothesis (premise)}$$

$$2- (r \vee s) \rightarrow \sim p \quad \text{Contrapositive Law on (1)}$$

$$3- p \vee \sim q \quad 2^{\text{nd}} \text{ hypothesis (premise)}$$

$$4- \sim p \rightarrow \sim q \quad \text{Implication Law on (3)}$$

$$5- (r \vee s) \rightarrow \sim q \quad \text{Inf. (2),(4)}$$

$$6- s \quad 3^{\text{rd}} \text{ hypothesis (premise)}$$

$$7- r \vee s \quad \text{Inf. (6)}$$

$$8- \sim q \quad \text{Inf. (5),(7)}$$

$$9- \sim q \wedge s \quad \text{Inf. (6),(8)}$$

(v) Given

$$(1) p \vee q$$

$$(2) q \rightarrow r$$

$$(3) \sim r$$

---

$$\therefore p$$

**Solution:**

1- $q \rightarrow r$	2 <sup>nd</sup> hypothesis(premise)
2- $\sim r \rightarrow \sim q$	Contrapositive Law on (1)
3- $\sim r$	3 <sup>rd</sup> hypothesis(premise)
4- $\sim q$	Inf. (2),(3)
5- $p \vee q$	1 <sup>st</sup> hypothesis(premise)
6- $(p \vee q) \wedge \sim q$	Inf. (4),(5)
7- $(p \wedge \sim q) \vee (q \wedge \sim q)$	Distributive Law on (6)
8- $(p \wedge \sim q) \vee F$	Contradiction Law (7)
9- $(p \wedge \sim q)$	Identity Law on (8)
10- $p$	Inf. (9) properties of $\wedge$

(vi) Given

- (1) If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on;
- (2) If the sailing race is held, then the cup will be awarded;
- (3) The cup was not awarded.

**Does this imply that: “It rained”?**

**Solution.**

p: rain;

q: foggy;

r: the sailing race will be held;

s: the lifesaving demonstration will go on;

t: then the cup will be awarded.

*Symbolically, the proposition is*

$$(1) \sim p \vee \sim q \rightarrow r \wedge s$$

$$(2) \quad \quad \quad r \rightarrow t$$

$$(3) \quad \quad \quad \sim t$$

p

1.  $\sim t$

3<sup>rd</sup> hypothesis

2.  $r \rightarrow t$

2<sup>nd</sup> hypothesis

3.  $\sim t \rightarrow \sim r$

Contrapositive of 2

4.  $\sim r$

Inf. (1),(3)

5.  $\sim p \vee \sim q \rightarrow r \wedge s$

1<sup>st</sup> hypothesis

6.  $\sim(r \wedge s) \rightarrow \sim(\sim p \vee \sim q)$

Contrapositive of 5

7.  $\sim r \vee \sim s \rightarrow (p \wedge q)$

De Morgan's law and double negation law on (5)

8.  $\sim r \vee \sim s$

Inf. (4) and domination law

9.  $p \wedge q$

Inf. (7),(8)

10. p

Inf. (9)

**Example 1.4.3.** Use the logical equivalences to show that

(i)  $\sim(p \rightarrow q) \equiv p \wedge \sim q$ ,

(ii)  $\sim(p \vee \sim(p \wedge q))$  is a contradiction,

(iii)  $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p \wedge \sim q)$ ,

(iv)  $p \vee (p \wedge q) \equiv p$  (Absorption Law).

**Solution.**

(i)  $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$  Implication Law



## 1.5. Logical Implication

**Definition 1.5.1. (Logical implication)**

We say the logical proposition “**r**” implies the logical proposition “**s**” (or **s** logically deduced from **r**) and write  $(r \implies s)$  iff  $(r \rightarrow s)$  is a tautology.

**Example 1.5.2.** Show that  $[(p \rightarrow t) \wedge (t \rightarrow q)] \implies (p \rightarrow q)$ .

**Solution.** Let P: the proposition  $(p \rightarrow t) \wedge (t \rightarrow q)$

Q: the proposition  $p \rightarrow q$

p	t	q	$p \rightarrow t$	$t \rightarrow q$	P	Q	$P \rightarrow Q$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

**Remark 1.5.3.**

(i) We use  $(r \implies s)$  to imply that the statement  $(r \rightarrow s)$  is true, while the statement  $(r \rightarrow s)$  alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for “**implies**”.

(ii) If  $(r \implies s)$  and  $(s \implies r)$ , then we denote that by  $(r \iff s)$ .

**Example 1.5.4.** Show that

(i)  $(r \implies s) \equiv [(\sim r \vee s)$  is tautology].

(ii)  $(r \iff s) \equiv (r \equiv s)$ .

**Solution.**

(i)

- 1-  $(r \implies s) \equiv (r \rightarrow s)$  is tautology      (Def. of  $\implies$ )
- 2-  $(r \rightarrow s) \equiv (\sim r \vee s)$       Logical Implication Law
- 3-  $(\sim r \vee s)$  is tautology      Inf. (1),(2)

(ii)

1-  $(r \Rightarrow s) \equiv (r \rightarrow s)$  is tautology } Def. of  $(\Rightarrow)$  and  $(\Leftrightarrow)$   
   $(s \Rightarrow r) \equiv (s \rightarrow r)$  is tautology }

2-  $(r \rightarrow s) \wedge (s \rightarrow r)$  is tautology.

3-  $r \leftrightarrow s$  is tautology

Equivalence Law

4-  $r \equiv s$ .

Inf. Remark 1.3.11

➤ Generally, the statement and its converse not necessary equivalent. Therefore,  $p \Rightarrow q$  does not mean that  $q \Rightarrow p$ .

**Example 1.5.5.** The statement “the triangle which has equal sides, has two equal legs” equivalent to the statement “ the triangle which has not two equal legs has no equal sides”.

## 1.6. Quantifiers

### Definition 1.6.1.

(i) A **predicate** or **propositional function** is a statement (formula) containing variables and that may be true or false depending on the values of these variables.

- That is, a predicate is a property or relationship between objects represented symbolically.
- We represent a predicate by a letter followed by the variables enclosed between parenthesis:  $P(x)$ ,  $Q(x, y)$ , etc.

(ii) An **example** for  $P(x)$  is: value of  $x$  for which  $P(x)$  is true.

(iii) A **counterexample**  $P(x)$  is: value of  $x$  for which  $P(x)$  is false.

(iv) The set,  $X$  which contain all possible value that satisfy the formula  $P$  is called a **universal set**.

(v) A set  $Y$  which contains all values  $x$  belong to set  $X$  such that  $P(x)$  is true is called a **solution set**.

$$Y = S_p = \{x \in X : P(x) \text{ is true}\}.$$

### Example 1.6.2.

(i)  $P(x) = x \leq 5 \wedge x > 3$  is true for  $x = 4$  and false for  $x = 6$  (counterexample).

(ii)  $P(x) = x \geq 5 \wedge x < 3$ , for every real numbers,  $x$  which is definitely false.

(iii) There exists an  $x$  such that  $(x) = x \leq 5 \vee x > 3$ , "which is definitely true.

(iv) Given the statement "Ahmad is a logician".

Let  $P$  represent 'is a logician' and let  $x$  represent 'Ahmad'. The predicate form of this statement is  $P(x)$ . That is,  $P(x) = \text{Ahmad is a logician}$ .

(v) Let  $r$ :  $x$  is married to  $y$ .

Let  $M$  represent "married". Then  $r = M(x, y)$ .

(vi) Let  $r$ : The numbers  $x$  and  $y$  are both odd.

This statement means  $(x \text{ is odd}) \wedge (y \text{ is odd})$ .

Let  $P$  represent 'is a odd' and let  $x, y$  represent 'numbers'. The predicate form of this statement is  $P(x) \wedge P(y)$ .

**Definition 1.6.3.**

(i)The phrase "**for all  $x$** " ("**for every  $x$** ", "**for each  $x$** ") is called a **universal quantifier** and is denoted by  $\forall x$ .

(ii)The phrase "**for some  $x$** " ("**there exists an  $x$** ") is called an **existential quantifier** and is denoted by  $\exists x$ .

**Definition 1.6.4. (The Universal Quantifier Proposition)**

Let  $f(x)$  be a proposition function which depend only on  $x$ . A sentence  $\forall x, f(x)$  read "For all  $x, f(x)$ " mean

"For all values  $x$  in  $X$ (universal set), the predicate  $f(x)$  is true."; that is,

$$\frac{\forall x, f(x)}{\therefore f(a)}$$

**Example 1.6.5.**

(i) r: The square of all real numbers are positive.

$$\boxed{r: \forall x \in \mathbb{R}, (x^2 > 0).}$$

(ii) r: The commutative law of addition of real numbers is hold.

$$\boxed{r: \forall x, \forall y \in \mathbb{R}, (x + y) = (y + x).}$$

(iii) r: The associative law of addition of real numbers is hold.

$$\boxed{r: \forall x, \forall y, \forall z \in \mathbb{R}, ((x + y) + z = x + (y + z)).}$$

(iv) r: All logicians are exceptional.

Let  $L$  represent '**set of logician**' and let  $E$  represent '**is exceptional**'. The predicate form of this statement is  $\boxed{r: \forall x \in L, E(x)}$ .

(v) r: All cars are red.

Let  $X :=$  **Set of cars**,  $f :=$  **is red**. Then,  $r : \forall x \in X, f(x)$ .



**Example 1.6.8.**

(i)  $\exists x: (x \geq x^2)$  is true since  $x = 0$  is a solution. There are many others.

(ii)  $r$ : Some logicians are exceptional.

Let  $L$  represent ‘**set of logician**’ and let  $E$  represent ‘**is exceptional**’. The predicate form of this statement is  $r: \exists x \in L, E(x)$ .

(iii)  $r$ : There is a car which is red.

Let  $X :=$  **Set of cars**,  $f :=$  **is red**. Then,  $r: \exists x \in X, f(x)$ .

**Remark 1.6.9.**

(i) The “**some**” form the existential quantifier, is frequently encountered in the following context:

$$\exists x(f(x) \wedge Q(x)),$$

which may be read,

**“Some  $x$  in a universal set  $X$  satisfying  $f(x)$  and satisfy  $Q(x)$ ”.**

For example:

(a)  $r$ : Some logicians are exceptional.

Let  $L$  represent ‘**is a logician**’ and let  $E$  represent ‘**is exceptional**’. Then

- Predicate Logic:  $r: \exists x(L(x) \wedge E(x))$
- In logical English: “For some  $x$ ,  $x$  is a logician and  $x$  is exceptional.”

(b)  $r$ : The square of some integers numbers are four (or: There is an integer for which its square is four)

Let  $P$  represent:  $\in \mathbb{Z}$  and let  $Q$  represent “**is 4**”.

- Predicate Logic:  $r: \exists x(P(x) \wedge Q(x))$ ; that is,

$$r: \exists x(x \in \mathbb{Z} \wedge x^2 = 4).$$

- In logical English: “For some  $x$ ,  $x$  is an integer number and  $x^2 = 4$ ”.

(c) At least one integer number is even (or: Some integers are even).

Let  $P$  represent:  $\in \mathbb{Z}$  and let  $E$  represent “**is even**”.

- Predicate Logic:  $r: \exists x(P(x) \wedge E(x))$ ; that is,

$$r: \exists x(x \in \mathbb{Z} \wedge E(x)).$$

- In logical English: “For some  $x$ ,  $x$  is an integer number and  $x$  is even.”

**Negation Rules of Quantifiers 1.6.10.**

(i) When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negate the statement.

(ii)  $\sim (x = y) = (x \neq y)$ .

(iii)  $\sim (x \equiv y) = (x \not\equiv y)$ .

(iv)  $\sim (x < y) = (y \leq x)$ .

(v)  $\sim (x \in Y) = (x \notin Y)$ .

(vi)  $\sim (\text{Even number}) = \text{Odd number}$ .

Now define the formal universal quantifier proposition using negation.

**Definition 1.6.11.**

(i)  $\forall x, f(x) \equiv (\sim \exists )x, \sim f(x)$ .

(ii)  $\exists x, f(x) \equiv (\sim \forall )x, \sim f(x)$ .

**Example 1.6.12.**

r: All logicians are exceptional.

Let  $L$  represent ‘set of logician’ and let  $E$  represent ‘is exceptional’.

- Predicate Logic:  $r: \forall x \in L, E(x) = (\sim \exists )x \in L, \sim E(x)$ .
- In logical English: “There is no  $x$  is a logician, for which  $x$  is not exceptional.”

**Equivalent Definitions 1.6.13.**

(i)  $\sim (\forall x, f(x)) \equiv \exists x, \sim f(x)$ .

(ii)  $\sim (\exists x, f(x)) \equiv \forall x, \sim f(x)$ .

(iii)  $\sim [\forall x (f(x) \rightarrow Q(x))] \equiv \exists x (f(x) \wedge \sim Q(x))$

$\equiv$  Some  $f(x)$  are not  $Q(x)$  (**Mathematical context**).

(iv)  $\sim (\exists x, (f(x) \wedge Q(x))) \equiv \forall x, \sim f(x) \vee \sim Q(x) \equiv \forall x (f(x) \rightarrow \sim Q(x))$

$\equiv$  No  $f(x)$  are  $Q(x)$  (**Mathematical context**).

**Example 1.6.14.**

(i) Express each of the following sentences in the form  $\forall x, P(x)$  and then give its negation in both cases  $\forall x, P(x)$  and in words.

r: The square of every real number is non-negative.



(a) “you can solve every problem we come up with”,

(b) “you can solve at least one problem we come up with”.

(ii) r: All students do not pay full tuition.

Here “r” could reasonably be interpreted as

(a) Not all students pay full tuition (Or: There exist some students do not pay full tuition).

(b) No students are pay full tuition (Or: There are no students are pay full tuition).

(iii) r: All integer numbers are not even.”

(a) Not all integer numbers are even.

(b) No integer numbers are even (Or: There are no even integers).

**Combined Quantifiers 1.6.16.** There are six ways in which the quantifiers can be combined when two variables are present:

(1)  $\forall x \forall y, f(x, y) \equiv \forall y \forall x, f(x, y)$  = For every  $x$ , for every  $y$ ,  $f(x, y)$ .

(2)  $\forall x \exists y, f(x, y) \equiv$  For every  $x$ , there exists a  $y$  such that  $f(x, y)$ .

(3)  $\forall y \exists x, f(x, y) \equiv$  For every  $y$ , there exists an  $x$  such that  $f(x, y)$ .

(4)  $\exists x \forall y, f(x, y) \equiv$  There exists an  $x$  such that for every  $y$ ,  $f(x, y)$ .

(5)  $\exists y \forall x f(x, y) \equiv$  There exists a  $y$  such that for every  $x$ ,  $f(x, y)$ .

(6)  $\exists x \exists y, f(x, y) \equiv \exists y \exists x, f(x, y) \equiv$  There exists an  $x$  such that there exists a  $y$ ,  $f(x, y)$ .

**Example 1.6.17.**

(i) r:  $\exists x \in \mathbb{R} \exists y \in \mathbb{R} : P(x, y) := (x^2 + y^2 = 2xy)$ . The proposition “ r “ is true since  $x = y = 1$  is one of many solutions.

(ii) s:  $\forall x \in \mathbb{R} \exists y \in \mathbb{R} : P(x, y) := (y^3 = x)$ . The proposition “ s “ is true since  $y = \sqrt[3]{x}$  is solution for  $P(x, y)$ .



## 1.7. Logical Reasoning

**Definition 1.7.1. (Arguments)**

An **argument** is a series of statements starting from hypothesis (premises/assumptions) and ending with the conclusion.

From the definition, an argument might be valid or invalid.

**Definition 1.7.2. (Valid Arguments) (Proofs)**

An argument is said to be **valid** if the hypothesis implies the conclusion; that is, if  $s$  is a statement implies from the statements  $s_1, s_2, \dots, s_n$ , then write as

$$s_1, s_2, \dots, s_n \mapsto s.$$

**Note 1.7.3.** In mathematics, the word **proof** is used to mean simply a valid argument. Many proofs involve more than two premises and a conclusion.

**Example 1.7.4.**

- (i) Let  $s_1$ : Some mathematicians are engineering;
- $s_2$  : Ali is mathematician;
- $s$  : Ali is engineering.

Show that the argument is valid or not.

**Solution.**

The argument  $s_1, s_2 \mapsto s$  is not valid, since not all mathematicians are engineering.

- (ii) Let  $s_1$ : There is no lazy student

$s_2$  : Ali is artist

$s_3$  : All artist are lazy

Find a conclusion  $s$  for the above premises making the argument  $s_1, s_2, s_3 \mapsto s$  is valid.

**Solution.**

Ali is-----.

**Remark 1.7.5.**

- (i) An argument  $s_1, s_2, \dots, s_n \mapsto s$

is valid if and only if  $(s_1 \wedge s_2 \wedge \dots \wedge s_n) \rightarrow s$  is tautology; that is,

$$(s_1 \wedge s_2 \wedge \dots \wedge s_n) \Rightarrow s.$$

- (ii) An argument does not depend on the truth of the premises or the conclusion but it just interested only in the question

**“Is the conclusion implied by the conjunction of the premises?”**



	$A_1$	$A_2$	$A_3$	$A_1 \wedge A_2 \wedge A_3$	$C$	$(A_1 \wedge A_2 \wedge A_3) \rightarrow C$
s	$p \wedge q$	$p \rightarrow I$	$s \rightarrow r$		$\sim s$	
T	T	F	T	F	F	T
F	T	F	T	F	T	T
T	T	T	F	F	F	T
<b>F</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b> ←
T	F	T	T	F	F	T
F	F	T	T	F	T	T
T	F	T	F	F	F	T
F	F	T	T	F	T	T
T	F	T	T	F	F	T
F	F	T	T	F	T	T
T	F	T	F	F	F	T
F	F	T	T	F	T	T
T	F	T	T	F	F	T
F	F	T	T	F	T	T
T	F	T	F	F	F	T
F	F	T	T	F	T	T







**Application Example:**  
**Cryptography (التشفير)**

A	1 0 1 0 0	Q	1 0 0 1 1
B	0 0 0 1 0	R	1 0 0 1 0
C	1 0 1 0 1	S	1 0 0 0 0
D	0 0 1 1 0	T	0 1 1 1 0
E	1 0 1 1 0	U	0 0 0 1 1
F	1 0 1 1 1	V	0 1 1 0 1
G	1 1 0 0 0	W	0 1 1 1 1
H	1 1 0 1 0	X	0 0 1 0 0
I	0 0 0 0 1	Y	0 1 1 0 0
J	1 1 0 0 1	Z	1 0 0 0 1
K	0 0 1 1 1	Space	1 1 1 1 1
L	0 1 0 1 1	0	1 1 0 1 1
M	0 1 0 1 0	1	1 1 1 0 0
N	0 1 0 0 1	2	1 1 1 0 1
O	0 1 0 0 0	3	1 1 1 1 0
P	0 0 1 0 1	4	0 0 0 0 0

**Key: 0 0 1 0 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1 0 0 0**

**Plaintext: GO HOME**

Plaintext	G	O		H	O	M	E
Code	11000	01000	11111	11010	01000	01010	10110
Key	00101	01100	11010	10111	10000	01010	11001
XOR							
Encryption	11101	00100	00101	01101	11000	00000	01111
Ciphertext	2	X	P	V	G	4	W

Ciphertext	2	X	P	V	G	4	W
Code	11101	00100	00101	01101	11000	00000	01111
Key	00101	01100	11010	10111	10000	01010	11001
XOR							
Decryption	11000	01000	11111	11010	01000	01010	10110
Plaintext	GOHOME						



**Q4:** Let

p = John is healthy;  
q = John is wealthy;  
r = John is wise.

Represent symbolically:

- (i) John is healthy and wealthy but not wise.
- (ii) John is not wealthy but he is healthy and wise.
- (iii) John is neither healthy nor wealthy nor wise.

**Q5:** Translate the sentences into propositional expressions:

**"Neither the fox nor the lynx can catch the hare if the hare is alert and quick."**

**Q6:** Represent as propositional expressions.

**"You can either (stay at the hotel and watch TV ) or (you can go to the museum and spend some time there)".**

**Q7:** Given a sentence **"If we are on vacation, we go fishing."** Then

- (i) translate the sentence into a logical expression,
- (ii) write the negation of the logical expression and translate the negation into English,
- (iii) write the converse of the logical expression and translate the converse into English,
- (iv) write the inverse of the logical expression and translate the inverse into English,
- (v) write the contrapositive of the logical expression and translate the contrapositive into English.

**Q8:** Write the contrapositive, converse and inverse of the expressions:

$$p \rightarrow q;$$

$$\sim p \rightarrow q;$$

$$q \rightarrow \sim p.$$

**Q9:** Determine whether the following arguments are valid or invalid:

**(i) Premises:**

(a) If I read the newspaper in the kitchen, my glasses would be on the kitchen table.

(b) I did not read the newspaper in the kitchen.

**Conclusion:** My glasses are not on the kitchen table.

**(ii) Premises:**

(a) If I don't study hard, I will not pass this course

(b) If I don't pass this course I cannot graduate this year.

**Conclusion:** If I don't study hard, I won't graduate this year.

**(iii) Premises:**

(a) You will get an extra credit if you write a paper or if you solve the test problems.

(b) You don't write a paper; however, you get an extra credit.

**Conclusion:** You have solved the test problems.

**Q10:** Find an expression equivalent to  $p \rightarrow q$  that uses only  $\wedge$  and  $\sim$ .

**Q11:** Negate the following sentences.

(i) The number  $x$  is positive, but the number  $y$  is not positive.

(ii) If  $x$  is prime, then  $\sqrt{x}$  is not a rational number.

(iii) For every prime number  $p$ , there is another prime number  $q$  with  $q > p$ .

- (iv) There exists a real number  $a$  for which  $a + x = x$  for every real number  $x$ .
- (v) Every even integer greater than 2 is the sum of two primes.
- (vi) The integer  $x$  is even, but the integer  $y$  is odd.
- (vii) At least one of the integers  $x$  and  $y$  is even.
- (viii) The numbers  $x$  and  $y$  are both odd.
- (ix) For every real number  $x$  there is a real number  $y$  for which  $y^3 = x$ .
- (x) I don't eat anything that has a face.

**Q12:** Write the following propositions with quantifiers and then give its negation with translations into words.

- (i) Some counting numbers are greater than five.
- (ii) Every element of set  $D$  is less than 7.
- (iii) Some elements of set  $D$  are less than 13.
- (iv) Every counting number greater than 4 is greater than 2.
- (v) Some counting numbers are even.
- (vi) Every counting number which is divisible by 2 is even.
- (vii) Every counting number is even or odd.
- (viii) For every  $x$  in  $D_x$  and for every  $y \in D_x$ ,  $x$  plus  $y$  less than 3.
- (ix) At least one politician isn't a logician.
- (x) Only no logicians are politicians.

**Q13:** Prove that  $\sqrt{2}$  is irrational using contradiction method.