Points in the Complex Plane: Definitions and Examples

1 Interior, Boundary, and Exterior Points

In the complex plane, a circle is defined as the set of all complex numbers z that are at a fixed distance (called the radius) from a fixed point (called the center).

Definition 1.1. *Let*

$$z = x + iy \quad \text{and} \quad z_0 = x_0 + iy_0$$

be complex numbers, where z_0 is the center of the circle. Then the circle with center z_0 and radius $r > 0$ is defined by:

$$|z - z_0| = r.$$

Example 1.2.

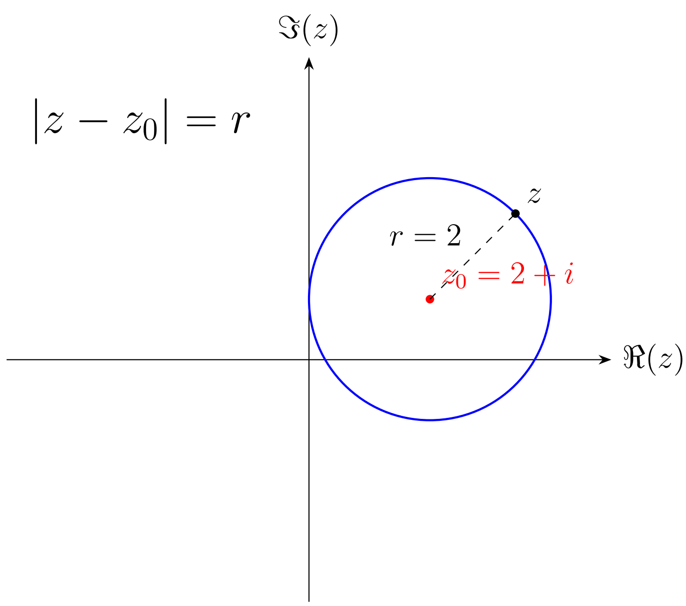
1- Unit circle:

$$|z| = 1$$

where Center $z_0 = 0$, radius $r = 1$.

2- Circle centered at $2 + i$ with radius 3:

$$|z - (2 + i)| = 3$$



In the complex plane, an open disk (or open circle region) is the set of all complex numbers whose distance from a fixed point z_0 (the center) is less than a given positive number r (the radius).

Definition 1.3. *Let $z, z_0 \in \mathbb{C}$ and $r > 0$. Then the open disk centered at z_0 with radius r is defined as:*

$$D(z_0, r) = z \in \mathbb{C} : |z - z_0| < r$$

1- Unit open disk:

$$D(0, 1) = z \in \mathbb{C} : |z| < 1.$$

This is the set of all complex numbers inside the unit circle, very important in complex analysis and geometric function theory.

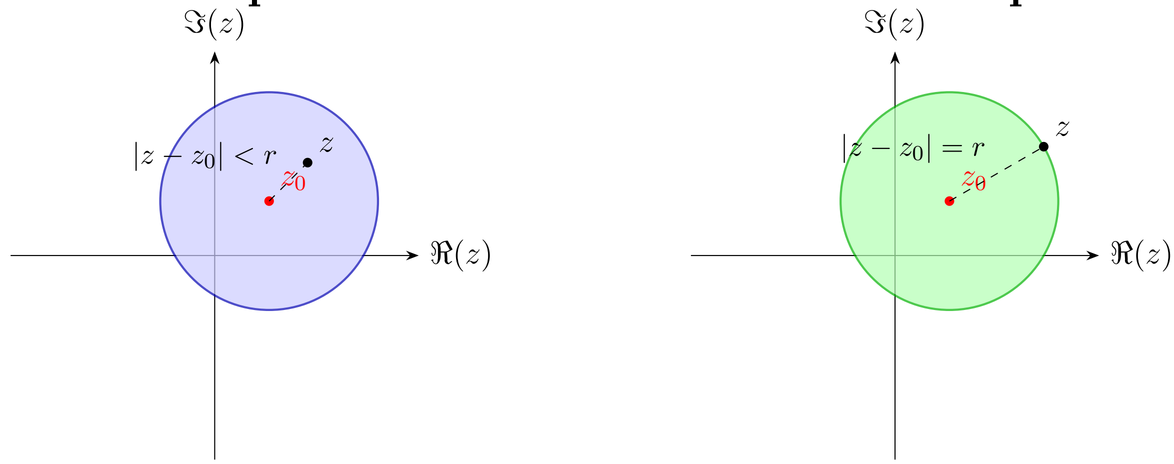
2- Open disk centered at $1 + i$ with radius 2:

$$D(1 + i, 2) = z : |z - (1 + i)| < 2$$

What is the relation between closed disk and open unite disk:

Open disk: $D(z_0, r) = \{|z - z_0| < r\}$, **and** Closed disk : $\overline{D}(z_0, r) = \{|z - z_0| \leq r\}$.

Comparison of Open and Closed Disks in the Complex Plane



Open Disk
$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$

Closed Disk
$\overline{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$

Definition 1.4. (Interior Point) Let $E \subset \mathbb{C}$ be a set in the complex plane, and let $z_0 \in E$. We say that z_0 is an interior point of E if there exists a small open disk centered at z_0 that lies entirely inside E.

Formally: A point $z_0 \in E$ is called an interior point of E if there exists a real number $r > 0$ such that

$$D(z_0, r) = z \in \mathbb{C} : |z - z_0| < r \subset E.$$

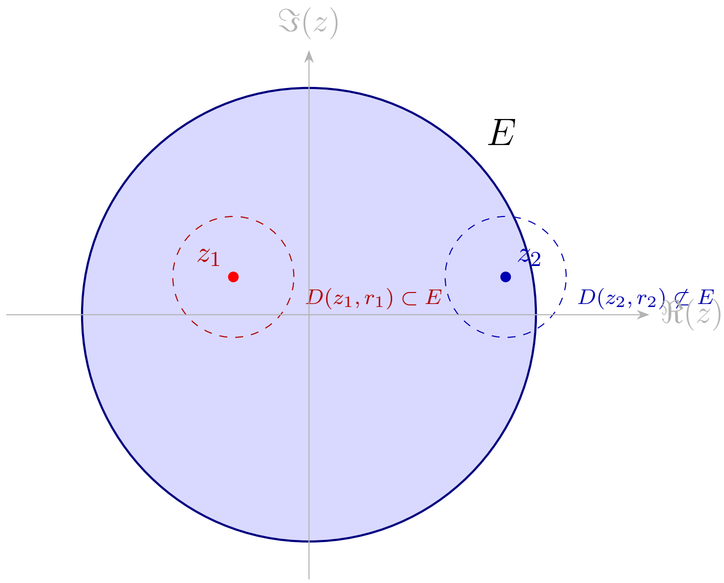
Example 1.5. *Let $E \subset \mathbb{C}$ be a set in the complex plane,*

1- *If $E = D(0,1) = z : |z| < 1$, then every point with $|z| < 1$ is an interior point of E.*

2- *The point $z_0 = 1$ is not an interior point of E, because any disk centered at 1 will extend outside the unit disk.*

Definition 1.6. (Set of all interior points:)
The set of all interior points of E is called the interior of E, denoted by:

$$E^\circ = z \in E : \exists, r > 0 \text{ such that } D(z,r) \subset E.$$



Interior and Non-Interior Points of a Set E in the Complex Plane

The following diagram that visually explains the interior point concept in the complex plane. It shows:

- a region E (a shaded disk),
- an interior point z_1 (red, with a small disk entirely inside E),
- and a non-interior point z_2 (blue, near the boundary).

Definition 1.7. (Boundary Point) Let $E \subset \mathbb{C}$ be a set in the complex plane. A point $z_0 \in \mathbb{C}$ is called a boundary point of E if every open disk centered at z_0 contains:
at least one point of E, and at least one point not in E.

Formally: A point z_0 is a boundary point of E if for every $r > 0$,

$$D(z_0, r) \cap E \neq \varnothing \quad \text{and} \quad D(z_0, r) \cap (\mathbb{C} \setminus E) \neq \varnothing.$$

Definition 1.8. ***(Set of all boundary points)*** *The boundary of E is the set of all such points:*

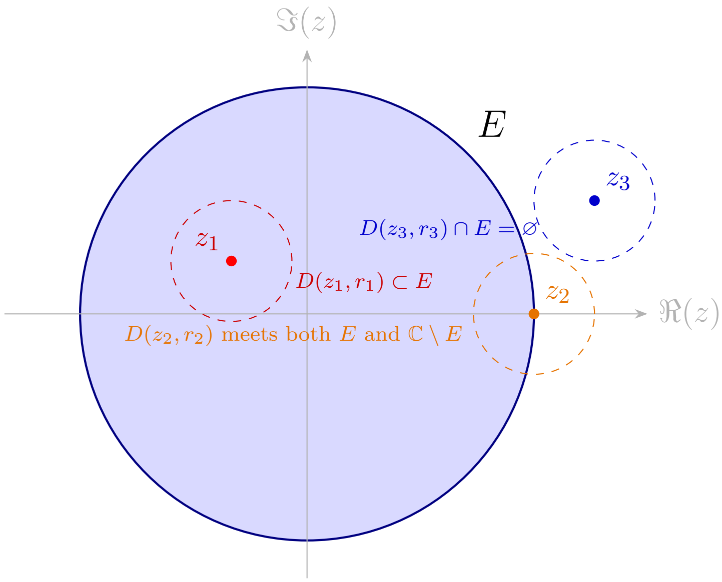
$$\partial E = z_0 \in \mathbb{C} : \forall r > 0, , D(z_0, r) \cap E \neq \varnothing \text{ and } D(z_0, r) \cap (\mathbb{C} \setminus E) \neq \varnothing.$$

Example 1.9. *Let $E = D(0,1) = z : |z| < 1$ (the open unit disk).*

- *Every point z with $|z| < 1$ is interior point.*
- *Every point z with $|z| = 1$ is boundary point.*
- *Every point z with $|z| > 1$ is exterior point.*

Hence:

$$\partial E = \{z : |z| = 1\}.$$



Interior, Boundary, and Exterior Points of a Set $E \subset \mathbb{C}$

Definitions: Let $E \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$.

1. **Interior Point:** z_0 is an interior point of E if there exists $r > 0$ such that

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset E.$$

2. **Boundary Point:** z_0 is a boundary point of E if for every $r > 0$,

$$D(z_0, r) \cap E \neq \varnothing \quad \text{and} \quad D(z_0, r) \cap (\mathbb{C} \setminus E) \neq \varnothing.$$

3. **Exterior Point:** z_0 is an exterior point of E if there exists $r > 0$ such that

$$D(z_0, r) \cap E = \varnothing.$$

Remark 1.10.

- *The set of all interior points of E is E°.*
- *The set of all boundary points of E is ∂E.*
- *Points that are neither interior nor boundary are exterior points.*

Definition 1.11. (Accumulation or Limit Point) Let $E \subset \mathbb{C}$ be a set in the complex plane. A point $z_0 \in \mathbb{C}$ is called an accumulation point (or limit point) of E if every open disk centered at z_0 contains at least one point of E different from z_0.

Formally: A point z_0 is an accumulation point of E if

$$\forall r > 0, \quad D(z_0, r) \cap (E \setminus z_0) \neq \varnothing.$$

Here, $D(z_0, r) = z \in \mathbb{C} : |z - z_0| < r$ is the open disk of radius r centered at z_0.

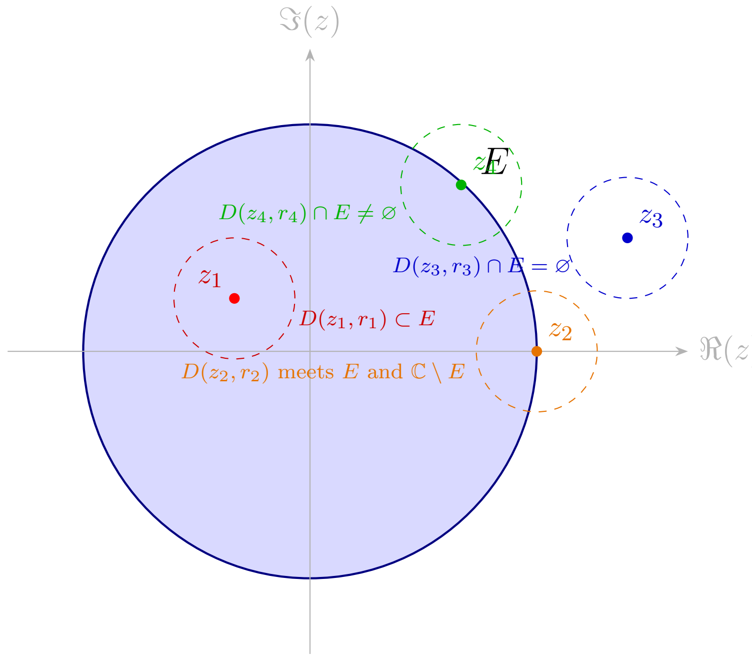
Example 1.12.

- Let, $E = \{1/n : n \in \mathbb{N}\} \subset \mathbb{R} \subset \mathbb{C}$, the point 0 is an accumulation point of E, because the numbers $1, 1/2, 1/3, \ldots$ get arbitrarily close to 0. No other points outside E are accumulation points in this example.
- Let, $E = D(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$, every point inside the open disk is an accumulation point. Every point on the unit circle $|z| = 1$ is also an accumulation point of E.

Notation: The set of all accumulation points of E is often denoted by

$$E' = \{z_0 \in \mathbb{C} : z_0 \text{ is an accumulation point of } E\}.$$

In the following a diagram showing interior points, boundary points, and accumulation points of a set E in the complex plane.



Interior, Boundary, Exterior, and Accumulation Points of $E \subset \mathbb{C}$

where z_1: Red is the interior point, disk lies entirely inside E. z_2: Orange is the boundary point, disk intersects both E and $\mathbb{C} \setminus E$. z_3: Green is the accumulation point, every disk contains points of E (other than itself).

Note that:
1- All interior points are accumulation points, but a boundary point may also be an accumulation point.
2- The green dashed circle shows a disk around the accumulation point intersecting E.
3- We can adjust the coordinates of z_3 to show interior, boundary, or outside accumulation points depending on the example.

Definition 1.13. ***(Isolated Point):*** *Let $E \subset \mathbb{C}$ be a set in the complex plane. A point $z_0 \in E$ is called an isolated point of E if there exists a small open disk centered at z_0 that contains no other points of E except z_0 itself.*

Formally:
A point $z_0 \in E$ is isolated if there exists $r > 0$ such that

$$D(z_0, r) \cap (E \setminus z_0) = \varnothing,$$

where $D(z_0, r) = z \in \mathbb{C} : |z - z_0| < r$.

Relation: Isolated points are **not accumulation points**, but accumulation points may or may not belong to E.

Example 1.14. *1. Let* $E = 1/n : n \in \mathbb{N} \subset \mathbb{R} \subset \mathbb{C}$*. Each point* $1, 1/2, 1/3, \ldots$ *is an isolated point of* E *except* 0*, which is an accumulation point.*
2. In the open unit disk $D(0,1) = z \in \mathbb{C} : |z| < 1$*. There are no isolated points, because every point has other points of the disk arbitrarily close.*

Relation to Accumulation Points:

The following example shows the isolated points are not accumulation points. Every accumulation point of E cannot be isolated.

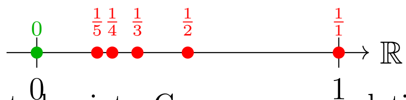
Example: Real Sequence

Let,

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

1) 0 is an accumulation point.

2) $1, 1/2, 1/3, \ldots$ are isolated points.



Red = isolated points, Green = accumulation point