**7. Normal Subgroups and Quotient Groups**

**Definition(7-1):**Let $(G,\*)$ be a group and $a,b\in G$, then $a$ is a conjugate to $b$ and denoted by $a\~b$ iff $∃x\in G\ni b=x\*a\*x^{-1}$ and $b\~a$ iff $∃x\in G\ni a=x\*b\*x^{-1}$.

$a≁b$ iff $b\ne x\*a\*x^{-1} ∀x\in G$

**Example(7-2):** In $(S\_{3},∘)$, is $f\_{3}\~f\_{2}$?

**Solution:** $x=f\_{1}⟹f\_{1}∘f\_{3}∘f\_{1}^{-1}=f\_{3}\ne f\_{2}$

$$x=f\_{2}⟹f\_{2}∘f\_{3}∘f\_{2}^{-1}=f\_{1}∘f\_{2}^{-1}=f\_{3}\ne f\_{2}$$

$$x=f\_{3}⟹f\_{3}∘f\_{3}∘f\_{3}^{-1}=f\_{2}∘f\_{2}=f\_{3}\ne f\_{2}$$

$$x=f\_{4}⟹f\_{4}∘f\_{3}∘f\_{4}^{-1}=f\_{5}∘f\_{4}=f\_{2}$$

$$x=f\_{5}⟹f\_{5}∘f\_{3}∘f\_{5}^{-1}=f\_{6}∘f\_{5}=f\_{2}$$

$$x=f\_{6}⟹f\_{6}∘f\_{3}∘f\_{6}^{-1}=f\_{4}∘f\_{6}=f\_{2}$$

$$⟹∃x\in S\_{3}\ni x∘f\_{3}∘x^{-1}=f\_{2}$$

$$⟹f\_{3}\~f\_{2}$$

Is $f\_{1}\~f\_{2}$ and $f\_{1}\~f\_{1}$? (**Homework**)

**Example(7-3):** In $(Z\_{4},+\_{4})$, is $1\~2$?

**Solution:** $x=1⟹1+\_{4}1+\_{4}1^{-1}=2+\_{4}3=5=1\ne 2$

$$x=2⟹2+\_{4}1+\_{4}2^{-1}=3+\_{4}2=5=1\ne 2$$

$$x=3⟹3+\_{4}1+\_{4}3^{-1}=4+\_{4}1=5=1\ne 2$$

$$x=0⟹0+\_{4}1+\_{4}0^{-1}=1\ne 2$$

$$⟹1≁2$$

**Remark(7-4):** If $(G,\*)$ is an abelian group and $a,b\in G$, then $a\~b⟺a=b$.

**Proof:** suppose that $a\~b⟺∃x\in G\ni b=x\*a\*x^{-1}$

$$⟺b=x\*x^{-1}\*a⟺b=a$$

**Theorem(7-5):** The relation (conjugate) is an equivalent relation.

**Proof:** (1) reflexive

let $a\in G$, to prove $a\~a$

$$∃e\in G\ni a=e\*a\*e^{-1}⟹a\~a$$

(2) symmetric

Let $a,b\in G$ and $a\~b$, to prove $b\~a$

$$a\~b⟹∃x\in G\ni b=x\*a\*x^{-1}$$

$$⟹x^{-1}\*b=a\*x^{-1}$$

$$⟹x^{-1}\*b\*x=a⟹b\~a$$

(3) transitive

Let $a,b,c\in G\ni a\~b$ and $b\~c$, to prove $a\~c$

$$a\~b⟹∃x\in G\ni b=x\*a\*x^{-1}…1$$

$$b\~c⟹∃y\in G\ni c=y\*b\*y^{-1}…2$$

Substitute $1$ in $2$, we get

$$c=y\*\left(x\*a\*x^{-1}\right)\*y^{-1}$$

$$c=\left(y\*x\right)\*a\*(y\*x)^{-1}$$

$c=z\*a\*z^{-1}$( where $z=y\*x \in G$)

$⟹a\~c$.

**Definition(7-6):** Let $(G,\*)$ be a group and $a\in G$, then the conjugate of$ a$ is denoted by $c(a)$ and defined as

 $c\left(a\right)=\{b\in G:a\~b\}$

or $c\left(a\right)=\{b\in G:a=x\*a\*x^{-1}\}$

or $c\left(a\right)=\{x\*a\*x^{-1}, ∀x\in G\}$

The set of all elements conjugate to $a$ is called the conjugate class of $a$.

**Examples(7-7):** Find the conjugate class of each element in the following groups:

1. $(S\_{3},∘)$ (**Homework**)
2. $(G\_{S},∘)$ (**Homework**)
3. $\left(G=\left\{1,-1,i,-i\right\},∙\right)\ni i^{2}=-1$.

**Solution:** $c\left(i\right)=\{x∙i∙x^{-1}, ∀x\in G\}$

$$=\{1∙i∙1^{-1}, -1∙i∙\left(-1\right)^{-1},i∙i∙i^{-1},-i∙i∙\left(-i\right)^{-1}\}$$

$$=\left\{i,i,i,i\right\}=\{i\}$$

$c\left(1\right)=\left\{1\right\}, c\left(-1\right)=\left\{-1\right\}, c\left(-i\right)=\{-i\}$.

**Example(7-8):** Find $c\left(3\right)$ in $(Z\_{4},+\_{4})$.

**Solution:** $c\left(3\right)=\{0+\_{4}3+\_{4}0^{-1}, 1+\_{4}3+\_{4}1^{-1},2+\_{4}3+\_{4}2^{-1},3+\_{4}3+\_{4}3^{-1}\}$

$=\{3\}$ (by Remark if$ G$ is an abelian group and$ a\~b$ , then $a=b$)

**Note(7-9):** Let $(G,\*)$ be a group and $ a\in G$, then $c(a)$ need not be a subgroup of $(G,\*)$, for example in $(S\_{3},∘)$, $c\left(f\_{3}\right)=\{f\_{2},f\_{3}\}$ is not a subgroup of $S\_{3}$.

**Theorem(7-10):** Let $(G,\*)$ be a group and $a,b\in G$, then

1. $a\in c\left(a\right) ∀a\in G$.

**Proof:** since $a\~a ∀a\in G$ ($\~$ is a reflexive)

$$a\in c(a)⟹c(a)\ne ∅$$

1. $c\left(a\right)=c\left(b\right)⟺a\~b ∀a,b\in G$.

**Proof:** $(⟹)$ suppose that $c\left(a\right)=c\left(b\right)$ , to prove $a\~b$

By $1, a\in c\left(a\right)=c(b)⟹a\in c(b)⟹a\~b$

$(⟸)$ suppose that $a\~b$, to prove $c\left(a\right)=c\left(b\right)$

This means $c\left(a\right)⊆c\left(b\right)$ and $c\left(b\right)⊆c(a)$

Let $x\in c\left(b\right)⟹x\~a$ and $a\~b⟹x\~b$

$$⟹x\in c\left(b\right)⟹c\left(a\right)⊆c\left(b\right)…1$$

Let $x\in c\left(b\right)⟹x\~b$ and $a\~b⟹x\~a$

$$⟹x\in c\left(a\right)⟹c\left(b\right)⊆c\left(a\right)…2$$

From $1, 2$, we get $c\left(a\right)=c\left(b\right)$

1. $c\left(a\right)∩c\left(b\right)=∅$ iff $a≁b$ (**Homework**)
2. $c\left(a\right)∩c\left(b\right)=∅$ or $c\left(a\right)=c\left(b\right)$ (**Homework**)
3. $b\in c\left(a\right)⟺c\left(a\right)=c(b)$

**Proof:** $(⟹)$ let $b\in c\left(a\right)⟹b\~a⟹c\left(a\right)=c(b)$ ( by Theorem)

$\left(⟸\right) c\left(a\right)=c(b)⟹a\~b⟹b\~a⟹b\in c(a)$.

1. $c\left(a\right)=\left\{a\right\} ∀a\in G⟺G$ is an abelian group.

**Proof:** $c\left(a\right)=\left\{a\right\} ∀a\in G⟺x\*a\*x^{-1}=a ∀a\in G$

$⟺x\*a=a\*x⟺G$ is an abelian group.

1. $c\left(a\right)=\left\{a\right\}⟺a\in C(G)$ (**Homework**)
2. $c\left(e\right)=\left\{e\right\}$ (**Homework**)

**Definition(7-11):** Let $(G,\*)$ be a group and $a\in G$, then the normalizer of $ a$ is denoted by $N(a)$ and defined as $N\left(a\right)=\{x\in G:x\*a=a\*x\}$.

**Example(7-12):** In $(Z\_{8},+\_{8})$. Find $N(3)$.

**Solution:** $N\left(3\right)=\left\{x\in Z\_{8}:x+\_{8}3=3+\_{8}x\right\}$

$$ =\left\{0,1,2,3,4,5,6,7\right\}=Z\_{8}$$

**Theorem(7-13):** Let $(G,\*)$ be a group and $a\in G$, then

1. $(N\left(a\right),\*)$ is a subgroup of $(G,\*)$.

**Proof:** $N\left(a\right)=\{x\in G:x\*a=a\*x\}⊆G$

Since $e\*a=a\*e⟹e\in N(a)⟹N(a)\ne ∅$

Closure: let $x,y\in N(a)$, to prove $x\*y\in N(a)$

$$x\in N\left(a\right)⟹x\*a=a\*x$$

$$y\in N\left(a\right)⟹y\*a=a\*y$$

$$\left(x\*y\right)\*a=x\*\left(y\*a\right)=x\*\left(a\*y\right)=\left(x\*a\right)\*y=\left(a\*x\right)\*y$$

$$=a\*\left(x\*y\right)⟹x\*y\in N(a)$$

Let $x\in N(a)$, to prove $x^{-1}\in N(a)$

Since $x\in N\left(a\right)⟹x\*a=a\*x⟹x\*a\*x^{-1}=a$

$⟹a\*x^{-1}=x^{-1}\*a⟹x^{-1}\in N(a)⟹(N\left(a\right),\*)$ is a subgroup.

1. $C\left(G\right)=∩N\left(a\right) ∀ a\in G$ (**Homework**)
2. $N\left(a\right)=G ∀a\in G⟺(G,\*)$ is an abelian.

**Proof:**$ (⟹)$suppose that $N\left(a\right)=G ∀a\in G$, to prove $G $is an abelian

$∀x\in G=N\left(a\right)⟹x\in N\left(a\right) ∀a\in G$

$$⟹x\in N\left(a\right) ∀x, a\in G⟹x\*a=a\*x ∀x,a\in G$$

$⟹(G,\*)$ is an abelian

$\left(⟸\right)$ suppose that $(G,\*)$ is an abelian, to prove $N\left(a\right)=G$

This means $N\left(a\right)⊆G$ and $G⊆N(a)$

$N(a)⊆G$ (by definition of $N(a)$)

To prove $G⊆N(a)$

Let $x\in G$ and $G$ is an abelian

$$⟹x\*a=a\*x ∀x,a\in G$$

$$⟹x\in N\left(a\right) ∀a\in G⟹G⊆N\left(a\right)⟹G=N\left(a\right) ∀a\in G$$

1. $N\left(a\right)=G⟺a\in G$ (**Homework**)
2. $c\left(a\right)=[G:N\left(a\right)]$

**Proof:** $c\left(a\right)=\{x\*a\*x^{-1}: ∀x\in G\}$

$$\left[G:N\left(a\right)\right]=\{x\*N\left(a\right),∀x\in G\} $$

Define $f: \left[G:N\left(a\right)\right]⟶c\left(a\right)\ni f\left(x\*N\left(a\right)\right)=x\*a\*x^{-1} ∀x\in G$

To prove $f$ is a map, $f$ is an one to one, $f$ is an onto (**Homework**)

1. If $(G,\*)$ is a finite group, then $\frac{O(G)}{O(c\left(a\right))}$

**Proof:** by $1⟹(N\left(a\right),\*)$ is a subgroup of $(G,\*)$

By Lagrange Theorem $⟹\frac{O(G)}{O(N\left(a\right))}$

$$O(G=O\left(N\left(a\right)\right)∙\left[G:N\left(a\right)\right]=O\left(N\left(a\right)\right)∙O(c\left(a\right))$$

$$⟹\frac{O(G)}{O(c\left(a\right))}$$

**Definition(7-14):** Let $\left(H,\*\right), (K,\*)$ are two subgroups of $(G,\*)$, then $H$ is a conjugate subgroup of $K$ iff $∃x\in G\ni K=x\*H\*x^{-1}$ and denoted by $H\~K$.

$$H≁K⟺K\ne x\*H\*x^{-1}∀x\in G$$

**Example(7-15):** In $\left(S\_{3},∘\right), H=\left\{f\_{1},f\_{6}\right\}, K=\left\{f\_{1},f\_{5}\right\}$. Is $H\~K$?

**Solution:** this means, $∃x\in S\_{3}\ni x∘H∘x^{-1}=K$?

$$x=f\_{1}⟹f\_{1}∘\left\{f\_{1},f\_{6}\}∘f\_{1}^{-1}=\{f\_{1}∘f\_{1}∘f\_{1}^{-1},f\_{1}∘f\_{6}∘f\_{1}^{-1}\right\}$$

$$=\left\{f\_{1},f\_{6}\right\}\ne K$$

$$x=f\_{2}⟹f\_{2}∘\left\{f\_{1},f\_{6}\}∘f\_{2}^{-1}=\{f\_{2}∘f\_{1}∘f\_{2}^{-1},f\_{2}∘f\_{6}∘f\_{2}^{-1}\right\}$$

$$=\left\{f\_{1},f\_{5}\right\}=K$$

$⟹∃x=f\_{2}\ni H\~K$.

**Example(7-16):** In $\left(Z\_{12},+\_{12}\right), H=\left\{0,4,8\right\}, K=\{0,3,6,9\}$. Is $H\~K$?

**Solution:** this means, $∃x\in Z\_{12}\ni x+\_{12}H+\_{12}x^{-1}=K$

$$x=1⟹1+\_{12}\left\{0,4,8\right\}+\_{12}1^{-1}=H\ne K$$

Since $x+\_{12}H+\_{12}x^{-1}=x+\_{12}x^{-1}+\_{12}H=H\ne K$

$⟹ H≁K$.

**Example(7-17):** In $\left(G\_{S},∘\right)$, let $H=\left\{r\_{1},r\_{4}\right\}, K=\left\{r\_{1},r\_{2}\right\}$. Is $H\~K$?

(**Homework**)

**Theorem(7-18):** Let $\left(H,\*\right), (K,\*)$ are two subgroups of $(G,\*)$ and $H\~K$, then $O\left(H\right)=O(K)$.

**Proof:** since $H\~K⟹∃x\in G\ni K=x\*H\*x^{-1}$

To prove $O\left(H\right)=O\left(K\right)=O(x\*H\*x^{-1})$

Define$f:\left(H,\*\right)⟶\left(x\*H\*x^{-1},\*\right)\ni f\left(h\right)=x\*h\*x^{-1}∀h\in H$

To prove $f$ is a map ?

Let $h\_{1}=h\_{2}$, to prove$ f(h\_{1})=f(h\_{2})$

Since $h\_{1}=h\_{2}⟹x\*h\_{1}\*x^{-1}=x\*h\_{2}\*x^{-1}⟹ f(h\_{1})=f(h\_{2})$

$⟹f$ is a map.

Is $f$ an one to one ? let $ f(h\_{1})=f(h\_{2})$

$$⟹x\*h\_{1}\*x^{-1}=x\*h\_{2}\*x^{-1}$$

$⟹h\_{1}=h\_{2}⟹f$ is an one to one.

Is $f$ an onto?$ R\_{f}=\left\{f\left(h\right): ∀h\in H\right\}=\{x\*h\*x^{-1}: ∀h\in H\}$

$=x\*H\*x^{-1}⟹f$ is an onto.

$⟹O\left(H\right)=O\left(x\*H\*x^{-1}\right)=O(K)$.

**Theorem(7-19):** Let $(H,\*)$ be a subgroup of $(G,\*)$ and $x\in G$, then $(x\*H\*x^{-1},\*)$ is a subgroup of$ (G,\*)$.

**Proof:** $e\in G$ and $e\*H\*e^{-1}=H\ne ∅⟹x\*H\*x^{-1}\ne ∅$

$$x\*H\*x^{-1}=\{x\*h\*x^{-1}:∀h\in H\}$$

Let $a,b\in x\*H\*x^{-1}$, to prove $a\*b^{-1}\in x\*H\*x^{-1}$

Let $a\in x\*H\*x^{-1}⟹a=x\*h\_{1}\*x^{-1}\ni h\_{1}\in H$

Let $b\in x\*H\*x^{-1}⟹b=x\*h\_{2}\*x^{-1}\ni h\_{2}\in H$

$$a\*b^{-1}=\left(x\*h\_{1}\*x^{-1}\right)\*( x\*h\_{2}\*x^{-1})^{-1}$$

$$=\left(x\*h\_{1}\*x^{-1}\right)\*(x\*h\_{2}^{-1}\*x^{-1})$$

$$=\left(x\*h\_{1}\right)\*\left(x^{-1}\*x\right)\*(h\_{2}^{-1}\*x^{-1})$$

$$x\*\left(h\_{1}\*h\_{2}^{-1}\right)\*x^{-1}\in x\*H\*x^{-1}$$

$⟹(x\*H\*x^{-1},\*)$ is a subgroup of $(G,\*)$.

**Note(7-20):** The relation of conjugate is equivalent relation on the set of all subgroups of $G$. (**Homework**)

**Definition(7-21):** Let $(H,\*)$ be a subgroup of $(G,\*)$, then the conjugate class of $H$ is denoted by $C(H)$ and define as

$$C\left(H\right)=\{x\*H\*x^{-1}:∀x\in G\}$$

**Example(7-22)**$ \left(S\_{3},∘\right), H=\{f\_{1},f\_{4}\}$**:**, find $C(H)$.

**Solution:**$ C\left(H\right)=\{x\*H\*x^{-1}:∀x\in S\_{3}\}$

$$=\{f\_{1}∘\{f\_{1},f\_{4}\}∘f\_{1}^{-1}, f\_{2}∘\{f\_{1},f\_{4}\}∘f\_{2}^{-1},…,f\_{6}∘\{f\_{1},f\_{4}\}∘f\_{6}^{-1}\}$$

$$=\{\left\{f\_{1},f\_{4}\right\},\left\{f\_{1},f\_{6}\right\},…,\left\{f\_{1},f\_{5}\right\}\}$$

**Example(7-23):** $(G=\left\{e,a,b,c\right\},\*), a^{2}=b^{2}=c^{2}=e$, is the four-Klien group.$G$ is an abelian, $H=\{e,a\}⊆G$, find $C(H)$.

**Solution:**$ C\left(H\right)=\{x\*H\*x^{-1}:∀x\in G\}$

$=\left\{x\*x^{-1}\*H:∀x\in G\right\}=H$.

**Deffinition(7-24):** Let $(H,\*)$ be a subgroup of $(G,\*)$, then the normalizer of $H$ is denoted by $N(H)$ and defined as

$$N\left(H\right)=\{x\in G:x\*H=H\*x\}$$

**Example(7-25):** The group $\left(G\_{S},∘\right), H=\{r\_{1},r\_{3}\}$, find $N(H)$.

**Solution:** $N\left(H\right)=\{x\in G\_{S}:x∘H=H∘x\}$

$$x=r\_{1}⟹r\_{1}∘H=H∘r\_{1}$$

$$x=r\_{2}⟹r\_{2}∘H=H∘r\_{2}$$

$$N\left(H\right)=\left\{r\_{1}, r\_{2}, r\_{3},r\_{4},h,v,D\_{1},D\_{2}\right\}=G\_{S}$$

**Examples(7-26):** Find$ C\left(H\right),N(H)$ to each of the following:

1. The group $\left(S\_{3},∘\right), H\_{1}=\left\{f\_{1},f\_{5}\right\}, H\_{2}=\left\{f\_{1},f\_{4}\right\}$. (**Homework**)
2. The group $\left(G\_{S},∘\right), H\_{1}=\left\{r\_{3},r\_{1},v,h\right\}, H\_{2}=\{r\_{1},D\_{1}\}$. (**Homework**)
3. The group $\left(Z\_{12},+\_{12}\right), H=\left\{0,4,8\right\}$. (**Homework**)

**Theorem(7-27):** Let $(H,\*)$ be a subgroup of $(G,\*)$, then

1. $(N\left(H\right),\*)$ is a subgroup of $(G,\*)$ containing $H$.

**Proof:** since $e\*H=H\*e⟹e\in N(H)\ne ∅$

$$N\left(H\right)=\{x\in G\ni x\*H=H\*x\}⊆G$$

Let $a,b\in N(H)$, to prove $a\*b^{-1}\in N(H)$

This means $\left(a\*b^{-1}\right)\*H=H\*(a\*b^{-1})$

Since $a\in N\left(H\right)⟹a\*H=H\*a$

$$b\in N\left(H\right)⟹b\*H=H\*b$$

$$b\*H\*b^{-1}=H⟹H\*b^{-1}=b^{-1}\*H⟹b^{-1}\in N(H)$$

$\left(a\*b^{-1}\right)\*H=a\*\left(b^{-1}\*H\right)=a\*(H\*b^{-1})$ $(b^{-1}\in N(H))$

$$=\left(a\*H\right)\*b^{-1}=\left(H\*a\right)\*b^{-1}=H\*(a\*b^{-1})$$

$⟹a\*b^{-1}\in N(H)⟹(N\left(H\right),\*)$ is a subgroup of $(G,\*)$

To prove $H⊆N(H)$

Let $a\in H⟹a\*H=H, H\*a=H⟹a\*H=H\*a$

$$⟹a\in N(H)⟹H⊆N(H)$$

1. If $(G,\*)$ is an abelian group, then $N\left(H\right)=G$.

**Proof:** suppose that $G$ is an abelian group, to prove $N\left(H\right)=G$

This means $N\left(H\right)⊆G, G⊆N(H)$

By definition of $N\left(H\right)⟹N\left(H\right)⊆G$

Let $x\in G⟹x\*H=H\*x⟹x\in N(H)⟹G⊆N(H)$

$$⟹G=N(H)$$

1. $O\left(C\left(H\right)\right)=O(\left[G:N\left(H\right)\right])$ (**Homework**)
2. If $(G,\*)$ is a finite group, then $\frac{O(G)}{O(C\left(H\right))}$

**Note(7-28):** If $N\left(H\right)=G$, then $(G,\*)$ is an abelian group. (**Homework**)

**Definition(7-29):** A subgroup $(H,\*)$ is called a self-conjugate iff $C\left(H\right)=H$, this means $x\*H\*x^{-1}=H ∀x\in G$.

**Example(7-30):** In $\left(S\_{3},∘\right), H\_{1}=\left\{f\_{1},f\_{2},f\_{3}\right\}, H\_{2}=\left\{f\_{1},f\_{5}\right\}$

$C\left(H\_{1}\right)=H\_{1}⟹H\_{1}$ is a self-conjugate

$C\left(H\_{2}\right)\ne H\_{2}⟹H\_{2}$ is not a self-conjugate.

**Definition(7-31):** A subgroup $(H,\*)$ is called a normal subgroup of $(G,\*)$ denoted by $H⊳G⟺H $is a self-conjugate

Or $H⊳G⟺x\*H\*x^{-1}=H ∀x\in G$

$$H⋫G⟺∃x\in G\ni x\*H\*x^{-1}\ne H$$

**Example(7-32):** The group $\left(G\_{S},∘\right), H=\left\{r\_{3},r\_{1},v,h\right\}$

$$C\left(H\right)=H⟹H⊳G\_{S}$$

**Example(7-33):** The group $\left(S\_{3},∘\right), H=\left\{f\_{1},f\_{5}\right\}$

$$C\left(H\right)\ne H⟹H⋫S\_{3}$$

**Example(7-34):** The group $\left(Z\_{4},+\_{4}\right), H=\left\{0,2\right\}$

$$C\left(H\right)=H⟹H⊳Z\_{4}$$

**Theorem(7-35):** Let $(H,\*)$ be a subgroup of $(G,\*)$, then

1. $H⊳G⟺x\*H=H\*x ∀x\in G$.

**Proof:** $H⊳G⟺x\*H\*x^{-1}=H ∀x\in G$

$$⟺x\*H=H\*x ∀x\in G$$

1. $H⊳G⟺N\left(H\right)=G$

**Proof:** $(⟹)$ suppose that $H⊳G$, to prove $N\left(H\right)=G$

This means $N\left(H\right)⊆G, G⊆N(H)$

$N\left(H\right)⊆G$ (by definition of $N\left(H\right)$)

To prove $G⊆N(H)$

Let $x\in G⟹x\*H=H\*x⟹x\in N\left(H\right)⟹G⊆N(H)$

$$⟹G=N(H)$$

$(⟸)$ suppose that $G=N(H)$, to prove $H⊳G$

$∀x\in G⟹x\in N\left(H\right)⟹x\*H=H\*x⟹H⊳G$ (by $1$ )

1. $H⊳G⟺c\left(a\right)⊆H ∀a\in H$

**Proof: :** $(⟹)$ suppose that $H⊳G$, to prove$c\left(a\right)⊆H ∀a\in H$

Since $H⊳G$ by definition $x\*H\*x^{-1}=H⟹x\*H\*x^{-1}⊆H$

$$c\left(a\right)=\{x\*a\*x:∀a\in H\}⊆H $$

$(⟸)$ suppose that $c\left(a\right)⊆H ∀a\in H$

To prove $H⊳G$, this means $x\*H\*x^{-1}=H$

Which is $x\*H\*x^{-1}⊆H, H⊆x\*H\*x^{-1}$

$$c\left(a\right)⊆H⟹x\*H\*x^{-1}⊆H…1$$

To prove $H⊆x\*H\*x^{-1}$

Let $b\in H⟹b=e\*b\*e$

$$b=\left(x\*x^{-1}\right)\*b\*\left(x\*x^{-1}\right)=x\*\left(x^{-1}\*b\*x\right)\*x^{-1}$$

$$b=x\*h\*x^{-1}\in x\*H\*x^{-1}$$

$$⟹H⊆x\*H\*x^{-1}…2$$

From $1,2$, we get $H=x\*H\*x^{-1}∀a\in G⟹H⊳G $

1. $H⊳G⟺\left(x\*H\right)\*\left(y\*H\right)=\left(x\*y\right)\*H ∀x,y\in G$

**Proof:** $(⟹)$ suppose that $H⊳G⟹H\*x=x\*H$

$$\left(x\*H\right)\*\left(y\*H\right)=\left(x\*H\*y\right)\*H=x\*\left(H\*y\right)\*H$$

$$=x\*\left(y\*H\right)\*H=\left(x\*y\right)\*\left(H\*H\right)=\left(x\*y\right)\*H$$

$(⟸)$ suppose that $H⋫G⟹∃x\in G\ni x\*H\*x^{-1}\ne H$

$$\left(x\*H\right)\*\left(x^{-1}\*H\right)\ne H\*H⟹\left(x\*x^{-1}\right)\*H\ne H$$

$⟹e\*H\ne H$, but this is contradiction $⟹H⊳G$

**Theorem(7-36):** Let $(G,\*)$ be a group, then

1. $\{e\}⊳G$ (**Homework**)
2. $G⊳G$ (**Homework**)
3. $C(G)⊳G$ (**Homework**)

**Theorem(7-37):** Every subgroup of an abelian group is a normal subgroup.

**Proof:** let $(G,\*)$ be an abelian group and $(H,\*)$ be a subgroup of$(G,\*)$,

to prove $x\*H\*x^{-1}=H ∀x\in G$

$x\*H\*x^{-1}=\left(x\*x^{-1}\right)\*H=e\*H=H⟹H⊳G$.

**Note(7-38):** The converse of above theorem is not true, for example

$$\left(G=\left\{\pm 1,\pm i,\pm j,\pm k\right\},∙\right)\ni i^{2}=j^{2}=k^{2}=-1$$

$$ij=k$$

$ji=-k⟹ij\ne ji⟹G$ is not an abelian.

The subgroups of $G$ are $\left\{1\right\},G,\left\{\pm 1\right\},\left\{\pm 1,\pm i\right\},\left\{\pm 1,\pm j\right\},\{\pm 1,\pm k\}$

**Theorem(7-39):** Let $(H,\*)$ be a subgroup of $\left(G,\*\right)\ni \left[G:H\right]=2$, then $H⊳G$.

**Proof:** since $\left[G:H\right]=2$, then there are two distinct left (right) cosets of $H$ in $G$.$ H, a\*H\ni a\in G-H$ (left cosets of$ H$ in $G$)

$H, H\*a\ni a\in G-H$ (right cosets of$ H$ in $G$)

$$H∪a\*H=G, H∩a\*H=∅…1$$

$$H∪H\*a=G, H∩H\*a=∅…2$$

If $a\in H⟹a\*H=H=H\*a⟹a\*H=H\*a ∀ a\in H$

If $a\in G-H⟹a\*H=G-H=H\*a⟹a\*H=H\*a ∀ a\in H$

$⟹a\*H=H\*a ∀ a\in G⟹H⊳G$.

**Note(7-40):** The converse of above theorem is not true, for example

$\left(G\_{S},∘\right), H=\left\{r\_{1},r\_{4}\right\},H⊳G\_{S}$, but $\left[G\_{S}:H\right]=4\ne 2$.

**Note(7-41):** If $H⊳G$, then $H∩K⋫G, (H\*K)⋫G$, where $H,K$ are two subgroups of the group $\left(G,\*\right)$.

Consider $\left(S\_{3},∘\right), H=\left\{f\_{1}\right\}⊳S\_{3}$ and $K=\{f\_{1},f\_{4}\}⋭S\_{3}$

$H\*K=\{f\_{1},f\_{4}\}⋭S\_{3}$, since $C\left(H\*K\right)\ne H\*K$.

$$\left(G\_{S},∘\right), H=\left\{r\_{1},r\_{3},h,v\right\},K=\{r\_{1},v\}$$

$H∩K=\{r\_{1},v\}⋭G\_{S}$, since $C\left(H\*K\right)\ne H\*K$

$H⊳G\_{S}, K⋭G\_{S}$.

**Definition(7-42):** A group $\left(G,\*\right)$ is called a simple group iff $G$ has no proper normal subgroup.

**Examples(7-43):**

1. The group $\left(S\_{3},∘\right)$ is not a simple, since $H=\{f\_{1},f\_{2},f\_{3}\}⊳S\_{3}$.
2. The group $\left(G\_{S},∘\right)$ is not a simple, since $H=\left\{r\_{1},r\_{3},h,v\right\}⊳G\_{S}$.
3. The group $\left(Z\_{6},+\_{6}\right)$ is not a simple, since $H=\left\{0,3\right\}⊳Z\_{6}$.
4. The group $\left(Z\_{3},+\_{3}\right)$ is a simple group, since $Z\_{3}$ has no proper subgroup.

**Definition(7-44):** Let $H⊳G$ and $\frac{G}{H}=\{x\*H:x\in G\}$. Define $⨂$ on $\frac{G}{H}$ as follows:$\left(x\*H\right)⨂\left(y\*H\right)=\left(x\*y\right)\*H ∀x,y\in G$, $(\frac{G}{H},⨂)$ is called a quotient group of $G$ by $H$.

**Theorem(7-45):** Let $H⊳G$, then $(\frac{G}{H},⨂)$ is a group.

**Proof:** $\frac{G}{H}=\{x\*H:x\in G\}$**,** since $e\*H=H\in \frac{G}{H}\ne ∅$

Closure: let $a\*H,b\*H\in \frac{G}{H}$, $\left(a\*H\right)⨂\left(b\*H\right)=\left(a\*b\right)\*H\in \frac{G}{H}$

Associative: let $a\*H,b\*H,c\*H\in \frac{G}{H}$

$$\left[\left(a\*H\right)⨂\left(b\*H\right)\right]⨂\left(c\*H\right)=[\left(a\*b\right)\*H]⨂(c\*H)$$

$$=\left(\left(a\*b\right)\*c\right)\*H=\left(a\*\left(b\*c\right)\right)\*H=(a\*H)⨂[\left(b\*c\right)\*H]$$

$$=(a\*H) ⨂[\left(b\*H\right)⨂\left(c\*H\right)]$$

Identity: $e\*H=H\in \frac{G}{H}$

$$\left(a\*H\right)⨂\left(e\*H\right)=\left(a\*e\right)\*H=a\*H ∀a\*H\in \frac{G}{H}$$

$$\left(e\*H\right)⨂\left(a\*H\right)=\left(e\*a\right)\*H=a\*H$$

$⟹e\*H$ is an identity element of $\frac{G}{H}$

Inverse: let $a\*H\in \frac{G}{H}$, to prove$ (a\*H)^{-1}=a^{-1}\*H$

$\left(a\*H\right)⨂\left(a^{-1}\*H\right)=\left(a\*a^{-1}\right)\*H=e\*H=H$

$$\left(a^{-1}\*H\right)⨂\left(a\*H\right)=\left(a^{-1}\*a\right)\*H=e\*H=H$$

$⟹∀a\*H\in \frac{G}{H}∃a^{-1}\*H\in \frac{G}{H}⟹(\frac{G}{H},⨂)$ is a group.

**Example(7-46):** In the group $\left(Z\_{6},+\_{6}\right), H=\{0,3\}$, find $\frac{Z\_{6}}{H}$ (if exist).

**Solution:** $H⊳Z\_{6}⟹\frac{Z\_{6}}{H}$exist

$$o+\_{6}H=H$$

$$1+\_{6}H=\{1,4\}$$

$$2+\_{6}H=\{2,5\}$$

$$3+\_{6}H=\left\{3,0\right\}=H$$

$$4+\_{6}H=\left\{4,1\right\}=1+\_{6}H$$

$$5+\_{6}H=\left\{5,2\right\}=2+\_{6}H$$

$$⟹\frac{Z\_{6}}{H}=\{H,1+\_{6}H,2+\_{6}H\}$$

$$O\left(\frac{Z\_{6}}{H}\right)=3$$

|  |  |  |  |
| --- | --- | --- | --- |
| $$⨂$$ | $$H$$ | $$1+\_{6}H$$ | $$2+\_{6}H$$ |
| $$H$$ | $$H$$ | $$1+\_{6}H$$ | $$2+\_{6}H$$ |
| $$1+\_{6}H$$ | $$1+\_{6}H$$ | $$2+\_{6}H$$ | $$H$$ |
| $$2+\_{6}H$$ | $$2+\_{6}H$$ | $$H$$ | $$1+\_{6}H$$ |

$⟹(\frac{Z\_{6}}{H},⨂)$ is a quotient group, $H$ is an identity.

$$(1+\_{6}H)^{-1}=1^{-1}+\_{6}H=5+\_{6}H=2+\_{6}H$$

$$(2+\_{6}H)^{-1}=2^{-1}+\_{6}H=4+\_{6}H=1+\_{6}H$$

**Example(7-47):** In the group $\left(Z\_{20},+\_{20}\right), H=\left〈5\right〉$, find $\frac{Z\_{20}}{H}$ (if exist).(**Homework**)

**Example(7-48):** In the group $\left(S\_{3},∘\right)$, $H=\{f\_{1},f\_{2},f\_{3}\}$, find $\frac{S\_{3}}{H}$ (if exist).

**Solution:** since $H⊳S\_{3}⟹\frac{S\_{3}}{H}$ exist

$$f\_{1}∘H=H$$

$$f\_{2}∘H=\left\{f\_{2},f\_{3},f\_{1}\right\}=H$$

$$f\_{3}∘H=\left\{f\_{3},f\_{1},f\_{2}\right\}=H$$

$$f\_{4}∘H=\left\{f\_{4},f\_{6},f\_{5}\right\}$$

$$f\_{5}∘H=\left\{f\_{5},f\_{4},f\_{6}\right\}=f\_{4}∘H$$

$$f\_{6}∘H=\left\{f\_{6},f\_{5},f\_{4}\right\}=f\_{4}∘H$$

$$⟹\frac{S\_{3}}{H}=\{H,f\_{4}∘H\}$$

But if $H=\left\{f\_{1}, f\_{4}\right\}, H⋫S\_{3}⟹\frac{S\_{3}}{H}$ is not exist.

**Theorem(7-49):** The quotient group of an abelian is an abelian.

**Proof:** suppose that $(G,\*)$ is an abelian group and $(H,\*)$ is a subgroup of $\left(G,\*\right)\ni H⊳G⟹\frac{G}{H}$ is a group

Let $a\*H, b\*H\in \frac{G}{H}⟹\left(a\*H\right)⨂\left(b\*H\right)=\left(a\*b\right)\*H$

$=\left(b\*a\right)\*H=\left(b\*H\right)⨂\left(a\*H\right)⟹(\frac{G}{H},⨂)$ is an abelian group.

**Theorem(7-50):** If $(G,\*)$ is a cyclic group, then $(\frac{G}{H},⨂)$ is a cyclic group.

**Proof:** suppose that $(G,\*)$ is a cyclic group, $H$ is a subgroup of $G$.

$⟹∃a\in G\ni G=\left〈a\right〉=\{a^{k}:k\in Z\}$, since $G$ is a cyclic$⟹G$ is an abelian

$⟹H⊳G⟹\frac{G}{H}$ is a group. To prove $\frac{G}{H}$ is a cyclic group, this means there is $a\*H\in \frac{G}{H}\ni \frac{G}{H}=\left〈a\*H\right〉=\{(a\*H)^{k}:k\in Z\}$, to prove

$\frac{G}{H}⊆\left〈a\*H\right〉, \left〈a\*H\right〉⊆\frac{G}{H} $, let $x\*H\in \frac{G}{H}⟹x\in G=\left〈a\right〉⟹x=a^{r},r\in Z$

$x\*H=a^{r}\*H=\left(a\*a\*…\*a\right)\*H(r$-times$)$

$=a\*H⨂…⨂a\*H(r$-times$)$

$$(a\*H)^{r}\in \left〈a\*H\right〉⟹x\*H\in \left〈a\*H\right〉⟹\frac{G}{H}⊆\left〈a\*H\right〉$$

To prove $\left〈a\*H\right〉⊆\frac{G}{H}$, let $y\*H\in \left〈a\*H\right〉$

$$y\*H=(a\*H)^{s}\ni s\in Z$$

$$y\*H=a^{s}\*H\in \frac{G}{H}⟹y\*H\in \frac{G}{H}⟹\left〈a\*H\right〉⊆\frac{G}{H}⟹\left〈a\*H\right〉=\frac{G}{H}$$

Therefore, $(\frac{G}{H},⨂)$ is a cyclic group.

**Note(7-51):** The converse of above theorem is not true, for example:

$\left(S\_{3},∘\right)$, $H=\{f\_{1},f\_{2},f\_{3}\}⊳S\_{3}⟹\frac{S\_{3}}{H}$ is a group, $\frac{S\_{3}}{H}=\{H, f\_{4}∘H\}$

$O\left(\frac{S\_{3}}{H}\right)=2$ (prime order), $\frac{S\_{3}}{H}$ is a cyclic group, but $\left(S\_{3},∘\right)$ is not a cyclic

$$\frac{S\_{3}}{H}=\left〈f\_{4}∘H\right〉=\left\{f\_{4}∘H,\left(f\_{4}∘H\right)^{2}\right\}=\{f\_{4}∘H,f\_{1}∘H=H\}$$

**Theorem(7-52):** Let $(G,\*)$ be a group and$(\frac{G}{C(G)},⨂)$ is a cyclic group, then $(G,\*)$ is an abelian group.

**Note(7-53):** The converse of this theorem is not true, for example:

$(G=\left\{e,a,b,c\right\},\*), a^{2}=b^{2}=c^{2}=e$, $G$ is an abelian (not a cyclic)

$C\left(G\right)=G⟹\frac{G}{C\left(G\right)}=\frac{G}{G}=\{e,a,b,c\}⟹\frac{G}{C\left(G\right)}$ is not a cyclic.

**Definition(7-54):** Let $(G,\*)$ be a group. If $a,b\in G$, then the commutator of $a,b$ is $\left[a,b\right]=a\*b\*a^{-1}\*b^{-1}$.

The commutator $\left[a,b\right]=e⟺a\*b=b\*a$, this means$ a,b$ are commute, the identity element $e=\left[e,e\right]$ is a commutator.

**Example(7-55):** In the group$\left(Z\_{4},+\_{4}\right)$.

$$\left[3,2\right]=3+\_{4}2+\_{4}3^{-1}+\_{4}2^{-1}=3+\_{4}2+\_{4}1+\_{4}2=0$$

**Example(7-56):** In the group$\left(Z,+\right)$.

$$\left[5,4\right]=5+4+5^{-1}+4^{-1}=5+4-5-4=0$$

**Note(7-57):** The commutator is an identity iff $(G,\*)$ is an abelian group.

**Definition(7-58):** Let $(G,\*)$ be a group, then the commutator subgroup of $(G,\*)$ denoted by $[G,G]$ is the collection of all the finite products of commutators in $G$.

$$\left[G,G\right]=\left\{\prod\_{}^{}\left[a\_{i},b\_{i}\right]:a\_{i},b\_{i}\in G\right\}=\{\left[a\_{1},b\_{1}\right]\*\left[a\_{2},b\_{2}\right]\*…\*\left[a\_{k},b\_{k}\right]\}$$

**Theorem(7-59):** The group $(\left[G,G\right],\*)$ is a normal subgroup of $(G,\*)$.

**Proof:** to prove $\left[G,G\right]$ is a subgroup of $G$.

 $\left[G,G\right]\ne ∅$, since $\left[e,e\right]\in \left[G,G\right],e\in G$

Let $x,y\in [G,G]$, to prove$ x\*y^{-1}\in [G,G]$

$$x=\left[a\_{1},b\_{1}\right]\*…\*\left[a\_{n},b\_{n}\right]$$

$$y=\left[c\_{1},d\_{1}\right]\*…\*\left[c\_{n},d\_{n}\right]$$

$$x\*y^{-1}=\left[a\_{1},b\_{1}\right]\*…\*\left[a\_{n},b\_{n}\right]\*(\left[c\_{1},d\_{1}\right]\*…\*\left[c\_{n},d\_{n}\right])^{-1}$$

$$=\left[a\_{1},b\_{1}\right]\*…\*\left[a\_{n},b\_{n}\right]\*\left[d\_{n},c\_{n}\right]\*…\*\left[d\_{1},c\_{1}\right]\in [G,G]$$

Thus, $x\*y^{-1}\in \left[G,G\right]⟹[G,G]$ is a subgroup of $G$.

To prove $[G,G]$ is a normal subgroup, let$ x\in G$

To prove $ x\*\left[G,G\right]\*x^{-1}⊆[G,G]$, let $ a\in x\*\left[G,G\right]\*x^{-1}$

$$a=x\*c\*x^{-1}, c\in \left[G,G\right]=x\*c\*x^{-1}\*e=x\*c\*x^{-1}\*c^{-1}\*c$$

$$=x\*c\*\left(x^{-1}\*c^{-1}\right)\*c=\left[x,c\right]\*c$$

Therefore, $a\in \left[G,G\right]⟹[G,G]$ is a normal subgroup of $G$.

**Theorem(7-60):** Let $(H,\*)$ be a normal subgroup of $G$, then $(\frac{G}{H},⨂)$ is an abelian iff $[G,G]⊆H$.

**Proof:** suppose that $a\*H, b\*H\in \frac{G}{H}$ and $\frac{G}{H}$ is an abelian

$$⟺\left(a\*b\right)\*H=\left(b\*a\right)\*H⟺H\*\left(a\*b\right)=H\*(b\*a)$$

$$⟺a\*b\*\left(b\*a\right)^{-1}\in H⟺\left[a,b\right]\in H$$

$⟺\left[G,G\right]⊆H ∀\left[a,b\right]\in \left[G,G\right],a,b\in G$.

**Corollary(7-61):** Prove that $(\frac{G}{[G,G]},⨂)$ is an abelian group. (**Homework**)